

# FINITELY MANY NEAR-COHERENCE CLASSES OF ULTRAFILTERS

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**ABSTRACT.** We answer Banach's and Blass' question [2, Question 31] whether there can be finitely many and more than one near-coherence classes of ultrafilters. We show that for any  $n \in [2, \omega)$  there is a forcing extension in which there are exactly  $n$  near-coherence classes of ultrafilters. Thus together with results of Banach, Blass [2] and [11] this completely settles the long standing question on the possible numbers of near-coherence classes of ultrafilters.

In our forcing extensions, one near-coherence class is represented by a  $P$ -point with character  $\aleph_1$ . The other  $n - 1$  near-coherence classes are represented by Ramsey ultrafilters with character  $\aleph_2$  that are not  $P_{\aleph_2}$ -points. For constructing such a forcing extension we combine Ramsey theory on the space of block sequences with Ramsey theory in Ellentuck spaces. We develop forcing techniques for destroying and resurrecting selective ultrafilters. For defining the iterands we extend Blass' work [6] to get Milliken–Taylor ultrafilters with particular properties. For analysing our forcing iteration we extend an iteration theorem by Shelah from [35, Ch. XVIII] by allowing that the definitions of the relations depend on the stage.

Our result has applications in analysis, operator theory and models of arithmetic: For  $n \geq 2$  is it consistent relative to ZFC that there are exactly  $n$  composants of  $\beta(\mathbb{R}^+) - \mathbb{R}^+$ ,  $n$  proper subideals of the ideal of compact operators in a Hilbert space such that the union of any two of them is the whole,  $n$  cofinality classes of short models of arithmetic.

## 1. INTRODUCTION

We first recall the definitions:

By a *filter over  $\omega$*  we mean a non-empty subset of  $\mathcal{P}(\omega)$  that is closed under supersets and under finite intersections and that does not contain the empty set. We call a filter *non-principal* if it contains all cofinite sets and we call it an ultrafilter if it is a maximal filter.

For  $B \subseteq \omega$  and  $f: \omega \rightarrow \omega$ , we let  $f[B] = \{f(b) : b \in B\}$  and  $f^{-1}[B] = \{n : f(n) \in B\}$ . The set of all infinite subsets of  $\omega$  is denoted by  $[\omega]^\omega$ . For  $\mathcal{B} \subseteq \mathcal{P}(\omega)$  we let

$$f(\mathcal{B}) = \{X : f^{-1}[X] \in \mathcal{B}\}.$$

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This double lifting is an important function from  $\mathcal{P}(\mathcal{P}(\omega))$  into itself. In analysis the special case of  $f$  being finite-on-one (that means that the preimage of each natural number is finite) is particularly useful, see e.g., [5].

Let  $\mathcal{F}$  be a non-principal filter over  $\omega$  and let  $f: \omega \rightarrow \omega$  be finite-to-one. Then also  $f(\mathcal{F})$  is a non-principal filter. It is the filter generated by  $\{f[X] : X \in \mathcal{F}\}$ . From now on we consider only non-principal filters and ultrafilters. Two filters  $\mathcal{F}$  and  $\mathcal{G}$  are *nearly coherent*, if there is some finite-to-one  $f: \omega \rightarrow \omega$  such that  $f(\mathcal{F}) \cup f(\mathcal{G})$  generates a filter. We also say to this situation that  $f(\mathcal{F})$  and  $f(\mathcal{G})$  are coherent. On the set of non-principal ultrafilters near coherence is an equivalence relation whose equivalence classes are called near-coherence classes. Near coherence is witnessed by a weakly increasing surjective finite-to-one function.  $f$  is weakly increasing if  $x < y \rightarrow f(x) \leq f(y)$ .

The purpose of this paper is to show the following:

**Main Theorem.** *Let  $n \in \omega \setminus \{0, 1\}$ . It is consistent relative to ZFC that there are exactly  $n$  near-coherence classes of ultrafilters.*

(If we want to consider near-coherence on the set of all ultrafilters over  $\omega$ , then there is the additional near-coherence class of all the principal ultrafilters.) Putting our result together with the results of Banach and Blass [2] and Blass [4] and [11] we get

**Corollary.** *“It is consistent relative to ZFC that there are exactly  $\kappa$  near-coherence classes of non-principal ultrafilters” is true exactly for any finite non zero  $\kappa$  and for  $\kappa = 2^{2^\omega}$ .*

By work of Mioduszewski our result has applications to analysis, namely the number of composants of  $\beta(\mathbb{R}^+) - \mathbb{R}^+$  corresponds by [30, 31] to the number of near-coherence classes of ultrafilters. Blass [4] gives applications to cofinality classes of short non standard models of arithmetic, and to the decomposition of the ideal of compact linear operators on a Hilbert space into proper subideals. His results on equivalent characterisations of indecomposability can be translated to: There is a decomposition into  $k$  proper subideals the union of any two different of them is the whole ideal of compact operators exactly when there are exactly  $k$  near-coherence classes. We indicate how the correspondence is defined (see [12]). Given an ideal associate to each positive selfadjoint operator  $A$  in it the sequence  $f \in \omega^\omega$  where  $f(n)$  is the number of eigenvalues  $\geq \frac{1}{2^n}$  of  $A$  counted with multiplicity. The sequences so obtained constitute a so-called shift ideal. To each proper shift ideal  $\mathcal{I}$  we chose a non-dominated function  $f$  and assign the filter  $\mathcal{F}_{\mathcal{I}} = \{\{n : g(n) \leq f(n)\} : g \in \mathcal{I}\}$ . The union of  $\mathcal{I}_1$  and  $\mathcal{I}_2$  generates a dominating shift ideal, so corresponds to the whole set of compact operators, iff  $\mathcal{F}_{\mathcal{I}_1}$  and  $\mathcal{F}_{\mathcal{I}_2}$  are non-nearly coherent filters.

A useful measure for growth is the  $\leq^*$ -relation: Let  ${}^\omega\omega$  denote the set of functions from  $\omega$  to  $\omega$ , and let  $f, g \in {}^\omega\omega$ . We say  $g$  eventually dominates  $f$

and write  $f \leq^* g$  if  $(\exists n)(\forall k \geq n)f(k) \leq g(k)$ . A family  $\mathcal{D} \subseteq {}^\omega\omega$  is called a dominating family iff  $\forall f \in {}^\omega\omega \exists g \in \mathcal{D} f \leq^* g$ .

**Definition 1.1.** Let  $E \in [\omega]^\omega$ . The function  $\text{next}(\cdot, E): \omega \rightarrow \omega$  is defined by  $\text{next}(n, E) = \min(E \cap [n, \infty))$ .

Now one key step in the mentioned translations from ultrafilter theory to the theory of operator ideals is the following:

**Fact 1.2.** (Proof of [5, Theorem 3.2]) Let  $\mathcal{V}, \mathcal{W}$  be ultrafilters over  $\omega$ .  $\mathcal{V}$  is nearly coherent to  $\mathcal{W}$  iff

$$\{\max(\text{next}(\cdot, X), \text{next}(\cdot, Y)) : X \in \mathcal{V}, Y \in \mathcal{W}\}$$

is not a  $\leq^*$ -dominating family.

The proof of the main theorem is the content of this paper. Besides the proof, there are three or four pages of “luxury” not used in the proof; namely in Section 2 we state and prove a sharp criterion for the existence of Milliken–Taylor ultrafilters with given projections. This result may be interesting for its own sake. Since the proof of the main theorem is long we indicate the passages that a reader who wants to focus on the main theorem can skip.

We give a short overview of the proof:  $P$ -points,  $Q$ -points, selective ultrafilters, selective coideals and Milliken–Taylor ultrafilters (see Def. 2.4) are involved in our forcing construction. We recall some definitions: We say “ $A$  is almost a subset of  $B$ ” and write  $A \subseteq^* B$  iff  $A \setminus B$  is finite. Similarly, the symbol  $=^*$  denotes equality up to finitely many exceptions in  $[\omega]^\omega$  or in  ${}^\omega\omega$ , the set of functions from  $\omega$  to  $\omega$ .

Let  $\kappa$  be a regular uncountable cardinal. An ultrafilter  $\mathcal{W}$  is called a  $P_\kappa$ -point if for every  $\gamma < \kappa$ , for every  $A_i \in \mathcal{U}$ ,  $i < \gamma$ , there is some  $A \in \mathcal{W}$  such that for all  $i < \gamma$ ,  $A \subseteq^* A_i$ ; such an  $A$  is called a *pseudo-intersection* of the  $A_i$ ,  $i < \gamma$ . A  $P_{\aleph_1}$ -point is called a  $P$ -point.

Let  $\mathbb{P}$  be a notion of forcing. We say that  $\mathbb{P}$  *preserves an ultrafilter*  $\mathcal{W}$  over  $I$  if

$$\Vdash_{\mathbb{P}} “(\forall X \subseteq I)(\exists Y \in \mathcal{W})(Y \subseteq X \vee Y \subseteq I \setminus X)”$$

and in the contrary case we say “ $\mathbb{P}$  destroys  $\mathcal{W}$ ”. In the first case  $\{X \in [\omega]^\omega \cap \mathbf{V}[G] : (\exists Y \in \mathcal{W}) X \supseteq Y\}$  is an ultrafilter in  $\mathbf{V}[G]$  and  $\mathcal{W}$  generates an ultrafilter in  $\mathbf{V}[G]$ . We just say:  $\mathcal{W}$  is an ultrafilter in  $\mathbf{V}[G]$ . If  $\mathbb{P}$  is proper and preserves  $\mathcal{W}$  and  $\mathcal{W}$  is a  $P$ -point, then  $\mathcal{W}$  stays a  $P$ -point [11, Lemma 3.2].

An ultrafilter  $\mathcal{W}$  is called a  $Q$ -point if for every increasing sequence  $\langle f_i : i \in \omega \rangle$  there is  $X \in \mathcal{W}$  such that for every  $i$ ,  $|X \cap [f_i, f_{i+1})| \leq 1$ . The  $P$ -point  $\mathcal{E}$  of our construction that can be initially also a  $Q$ -point will show that preservation of an ultrafilter does not imply preservation of its  $Q$ -property.

An ultrafilter  $\mathcal{R}$  is called *selective* (or *Ramsey ultrafilter*) if it is a  $P$ -point and a  $Q$ -point. We use the von Neumann natural numbers  $n = \{0, \dots, n-1\}$ . We often use the following, equivalent, characterisation of selectivity:

- (\*) For any  $\subseteq$ -descending sequence  $\langle A_n : n \in \omega \rangle$  of sets  $A_n \in \mathcal{R}$  there is  $A \in \mathcal{R}$  such that  $(\forall n \in \omega)(A \setminus (n+1) \subseteq A_n)$ .

Mathias introduced the following notion under the name “happy family” [21, Def. 0.1.]. Louveau studied it in the special case of ultrafilters [20]. Todorćević [38, Chapter 7] presents the topic under the name “local Ellentuck topology” and uses the name “selective coideal”.

**Definition 1.3.** (See [21, Def. 0.1.]) A set  $\mathcal{H} \subseteq [\omega]^\omega$  is called a selective coideal if the following hold:

- (i)  $\mathcal{I}_{\mathcal{H}} := \mathcal{P}(\omega) \setminus \mathcal{H}$  is an ideal that contains all singletons.
- (ii) If  $\langle A_i : i \in \omega \rangle$  is a  $\subseteq$ -descending sequence of elements  $A_i \in \mathcal{H}$ , then there is  $B \in \mathcal{H}$  such that  $(\forall i \in \omega) B \setminus (i+1) \subseteq A_i$ . We call such a  $B$  a diagonal pseudointersection.

We write  $\mathcal{I}_{\mathcal{H}}^* = \{\omega \setminus X : X \in \mathcal{I}_{\mathcal{H}}\} = \mathcal{F}_{\mathcal{H}}$ .  $\mathcal{F}_{\mathcal{H}}$  is a filter that is uniquely determined by  $\mathcal{H}$ . Then  $\mathcal{H}$  coincides with the  $\mathcal{F}_{\mathcal{H}}$ -positive sets, i.e.,

$$\mathcal{H} = \mathcal{F}_{\mathcal{H}}^+ := \{X \in [\omega]^\omega : (\forall Y \in \mathcal{F}_{\mathcal{H}})(X \cap Y \neq \emptyset)\}.$$

Also in general, given a filter  $\mathcal{F}$ , we let

$$\mathcal{F}^+ = \{X \in [\omega]^\omega : (\forall Y \in \mathcal{F})(Y \cap X \neq \emptyset)\}.$$

For a non-principal  $\mathcal{F}$ ,  $\mathcal{F}^+$  coincides with  $\{X \in [\omega]^\omega : (\forall Y \in \mathcal{F})(Y \cap X \in [\omega]^\omega)\}$ .

We give an outline of the construction: We fix  $n \geq 2$ , the number of classes we are aiming for. We let  $S_i^2 = \{\alpha \in \omega_2 : \text{cf}(\alpha) = \omega_i\}$ ,  $i = 0, 1$ . A ground model with CH and  $\diamond(S_1^2)$  is extended by a countable support iteration  $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \omega_2, \alpha < \omega_2 \rangle$ , with iterands of the form  $\mathbb{Q}_\alpha = \text{M}(\mathcal{U}_\alpha)$  (see Def. 3.2). The iterand  $\mathbb{Q}_\alpha$  is a  $\mathbb{P}_\alpha$ -name and  $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$ , and at limits we build  $\mathbb{P}_\alpha$  with countable supports. Alternatively, instead of assuming a diamond in the ground model we could force in a first forcing step with approximations  $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \gamma, \alpha < \gamma \rangle$ ,  $\gamma < \omega_2$ . For such a representation of a forcing see [26, Section 2].

The stratification of the universe given by iterated forcing gives rise to the following definitions:

**Definition 1.4.** (a)  $\mathbf{V}_\alpha = \mathbf{V}^{\mathbb{P}_\alpha}$  stands for an arbitrary  $\mathbb{P}_\alpha$ -generic extension of  $\mathbf{V} = \mathbf{V}_0$ .

- (b) Let  $\mathcal{B} \in \mathbf{V}_\alpha$  be a filter base in  $\mathbf{V}_\alpha$ , that is any intersection of finitely many members of  $\mathcal{B}$  is infinite. We let for  $\omega_2 \geq \beta \geq \alpha$

$$\text{fil}(\mathcal{B})^\beta = \{X \in [\omega]^\omega \cap \mathbf{V}_\beta : \exists B_0, \dots, B_n \in \mathcal{B} X \supseteq B_0 \cap \dots \cap B_n\},$$

the filter generated by  $\mathcal{B}$  in  $\mathbf{V}_\beta$ .

- (c) For  $\mathcal{F} \in \mathbf{V}_\alpha$  a filter,  $\beta \geq \alpha$ , we let

$$\mathcal{F}^{+\beta} = \{X \in \mathbf{V}_\beta \cap [\omega]^\omega : (\forall Y \in \mathcal{F})(X \cap Y \in [\omega]^\omega)\}.$$

The iterands  $\mathbb{Q}_\alpha$  are proper, indeed  $\sigma$ -centred iterands can be chosen. Their definition is based on ultrafilters and other names that are defined by induction. The essential properties of the iteration are:

- (1) Recall, Blass [4] showed that: If there are fewer than  $2^{\mathfrak{u}}$  near-coherence classes then  $\mathfrak{u} < \mathfrak{d}$ . Here  $\mathfrak{u}$  is the minimum character of a non-principal ultrafilter over  $\omega$ , and  $\mathfrak{d}$  is the dominating number. The character of an ultrafilter is the minimal size of a basis. A subset of an ultrafilter  $\mathcal{W}$  is called a basis of  $\mathcal{W}$  if every set in  $\mathcal{W}$  has a subset that is in the basis.

In our construction, one  $P$ -point  $\mathcal{E} \in \mathbf{V}_0$  that will be preserved throughout the iteration, and we destroy many others.  $\mathcal{E}$  witnesses  $\mathfrak{u} = \aleph_1$ . This is like in [25]. The extension of Eisworth's work [15] to Matet forcing with  $(k$ -coloured block)-sequences is the tool for preserving  $\mathcal{E}$ . During our construction  $\mathcal{E}$  will stay an ultrafilter over  $\omega$  (in the end it will be a  $P$ -point, but not a  $Q$ -point even if it was a  $Q$ -point in the ground model) and will be the representative of one near-coherence class. In the final evaluation of  $\mathbb{P}_{\omega_2}$ , moreover,  $\mathcal{E}$  with its small character will be responsible that any near-coherence class but the class of  $\mathcal{E}$  can be computed by a small set of functions.

- (2) Similar to [25, 24] we get “there are not more than  $n$  near-coherence classes” with the aid of a diamond. A diamond sequence on  $S_1^2$  is a sequence  $\langle \mathcal{D}_\alpha : \alpha \in S_1^2 \rangle$  such that for any  $X \subseteq \omega_2$  the set  $\{\alpha \in S_1^2 : X \cap \alpha = \mathcal{D}_\alpha\}$  is stationary. If the diamond  $\mathcal{D}_\alpha$  guesses an ultrafilter  $\mathcal{D}_\alpha[G] = \mathcal{W}$  that is not nearly coherent to any of the  $\mathcal{R}_{i,\alpha}$ , we add with  $\mathbb{Q}_\alpha$  a Matet real  $\mu_\alpha$  and  $f_{\mu_\alpha}$  with  $f_{\mu_\alpha}(n) = |\mu_\alpha \cap n|$ , such that  $f_{\mu_\alpha}(\mathcal{W}) = f_{\mu_\alpha}(\mathcal{E})$ . A reflection argument for countable support iterations of proper iterands of size  $\leq \alpha_1$  and the existence of a small set of test functions imply that there are at most  $n$  near-coherence classes.
- (3) Selective ultrafilters  $\mathcal{R}_{i,\alpha}$ ,  $i = 1, \dots, n-1 = k$ ,  $\alpha \leq \omega_2$ , will grow with  $\alpha$ , witnessing the  $n-1$  other classes. This task be our main new work. The tricky part of our proof is to find suitable (names for) Milliken–Taylor ultrafilters  $\mathcal{U}_\alpha$  over  $\mathbb{F}^k$  and selective ultrafilters  $\mathcal{R}_{i,\alpha} \in \mathbf{V}_\alpha$ ,  $i = 1, \dots, k = n-1$ , such that for any  $\alpha \leq \omega_2$

$$\begin{aligned}
 (1.1) \quad & \mathbb{P}_\alpha \Vdash (\forall i \in \{1, \dots, k\}) (\forall \gamma \leq \alpha) \left( \mathcal{R}_{i,\gamma} \cup \{\mu_\gamma(i)\} \subseteq \mathcal{R}_{i,\alpha} \right. \\
 & \quad \wedge \mathcal{R}_{i,\alpha} \text{ is a selective ultrafilter} \Big) \wedge \\
 & \quad \text{for } i \neq j, i, j \in \{1, \dots, k\}, \\
 & \quad \mathcal{R}_{i,\alpha} \text{ is not nearly coherent to } \mathcal{R}_{j,\alpha}.”
 \end{aligned}$$

and for any  $\alpha < \omega_2$

$$\begin{aligned}
 (1.2) \quad & \mathbb{P}_\alpha \Vdash “\mathcal{U}_\alpha \text{ is an Milliken–Taylor ultrafilter over } \mathbb{F}^k \wedge \\
 & \quad \Phi(\mathcal{U}_\alpha(i)) \subseteq \mathcal{R}_{i,\alpha} \wedge \Phi(\mathcal{U}_\alpha) \not\leq_{\text{RB}} \mathcal{E} \wedge \mathbb{Q}_\alpha = \mathbb{M}(\mathcal{U}_\alpha)” .
 \end{aligned}$$

The core  $\Phi(\mathcal{U})$  and the core of colour  $i$ ,  $\Phi(\mathcal{U}(i))$ , is defined in Def. 2.9, the Rudin–Blass ordering  $\leq_{\text{RB}}$  will be defined in Def. 2.11, the names

$\mu_\gamma(i)$  are defined in Def. 4.3. This ends our first coarse outline of the forcing extensions.

Now the main tasks for establishing these properties are ordered according to sections and read as follows:

In Section 2 we extend Blass' work [6] from  $\mathbb{F}$  to  $\mathbb{F}^k$ , and bear also on Blass' recent work [7]. The main result of Section 2 is:

**Theorem 2.12** *Assume CH. Given selective ultrafilters  $\mathcal{R}_1, \dots, \mathcal{R}_k$  and  $\mathcal{S}_1, \dots, \mathcal{S}_k$  such that each  $\mathcal{R}_i$  is not nearly coherent to any  $\mathcal{S}_j$  and given two ultrafilters  $\mathcal{E}, \mathcal{W}$  that are not nearly coherent to any of the  $\mathcal{R}_i, \mathcal{S}_j$ , we can find in an Milliken–Taylor ultrafilter  $\mathcal{U}$  over  $\mathbb{F}^k$  such that for  $i = 1, \dots, k$ ,  $\hat{\min}(\Phi(\mathcal{U}(i))) = \mathcal{R}_i$ ,  $\hat{\max}(\Phi(\mathcal{U}(i))) = \mathcal{S}_i$  and  $\Phi(\mathcal{U}) \not\leq_{\text{RB}} \mathcal{E}, \mathcal{W}$ .*

There are no requirements on non-near coherence among the  $\mathcal{R}_i$  nor among the  $\mathcal{S}_j$ . We show that the premise of this theorem is as weak as possible. In the applications  $\mathcal{E}$  will be our initial  $P$ -point, and  $\mathcal{W}$  will appear only at successor steps  $\alpha \mapsto \alpha + 1$ , as an ultrafilter in  $\mathbf{V}_\alpha$  that is handed down by a  $\diamond(S_1^2)$ -sequence.

In Section 3 we explain Matet forcing with centred systems and with Milliken–Taylor ultrafilters and recall Eisworth's work on the preservation of  $P$ -points and extend it to forcings with Milliken–Taylor ultrafilters with  $k$  colours.

In Section 4 we complement the selective ultrafilters  $\mathcal{R}_{i,\alpha}$  from the models  $\mathbf{V}_\alpha$  that are destroyed in  $\mathbf{V}_{\alpha+1}$  to a selective ultrafilter  $\mathcal{R}_{i,\alpha+1} = \text{fil}(\mathcal{R}_{i,\alpha} \cup \{\mu_\alpha(i)\})^{\alpha+1}$  and for a limit  $\alpha$  of countable cofinality we find selective ultrafilters  $\mathcal{R}_{i,\alpha} \supseteq \bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma}$ . In this extension process we preserve that the respective  $\mathcal{R}_{i,\alpha}$  are pairwise not nearly coherent and not nearly coherent to  $\mathcal{E}$ .

For this we prove preservation theorems in the style of [35, Ch XVIII] for a particular relation  $\bar{R}$  (see Def. 4.10) about selective ultrafilters and about a selective coideal in the forcing extension. The technical highlights will be Theorem 4.5, Theorem 4.18, Corollary 4.21. The new feature in comparison to [35, Ch XVIII] is that the definition of the relation  $\bar{R}$  and the "coverers"  $g_{\alpha,a}$  depend on the iteration stage.

One clause in the definition of  $\bar{R}$  will be a relation Mathias used in his proof of

**Proposition** [21, Prop. 011] *Assume CH. If  $\mathcal{H}$  is a selective coideal then there is a selective ultrafilter  $\mathcal{R} \subseteq \mathcal{H}$ .*

Our Proposition 4.20 is an adaption of Mathias' proposition and its proof into our forcing environment.

In Section 5 we define iterated forcings that establish our consistency results. In Section 6, we use our forcing to answer the last question in Blass' list of [6].

Undefined notation on cardinal characteristics can be found in [3, 8]. Undefined notation about forcing can be found in [19, 35]. In the forcing,

we follow the Israeli style that the stronger condition is the *larger* one. For Sections 3 to 5 very good background in proper forcing is assumed.

## 2. MILLIKEN–TAYLOR ULTRAFILTERS WITH GIVEN MINIMUM AND MAXIMUM PROJECTIONS

The aim of the section is to provide the Milliken–Taylor ultrafilters  $\mathcal{U}_\alpha$  over  $\mathbb{F}^k$  for the definition of the iterand  $\mathbb{Q}_\alpha$  under the assumption that selective ultrafilters  $\mathcal{R}_i, \mathcal{S}_j, i, j \in \{1, \dots, k\}$  and two ultrafilters  $\mathcal{E}, \mathcal{W}$  are given.

We first introduce some notation about block-sequences. Our nomenclature follows Blass [6], Eisworth [15] and Todorćević [38].

**Definition 2.1.** *Let  $k \geq 1$ .*

(1) *We let*

$$\mathbb{F}^k = \{a : a : \text{dom}(a) \rightarrow \{1, \dots, k\}, \\ \text{dom}(a) \subseteq \omega \text{ is finite} \wedge (\forall i = 1, \dots, k)(a^{-1}[\{i\}] \neq \emptyset)\}.$$

(2)  *$a \in \mathbb{F}^k$  is called a block.*

(3) *For  $a, b \in \mathbb{F}^k$  we write  $a < b$  if  $(\forall n \in \text{dom}(a))(\forall m \in \text{dom}(b))(n < m)$ .*

(4) *A sequence  $\bar{a} = \langle a_n : n \in \omega \rangle$  of members of  $\mathbb{F}^k$  is called unmeshed if for all  $n$ ,  $a_n < a_{n+1}$ . By  $(\mathbb{F}^k)^\omega$  we denote the set of unmeshed sequences of members in  $\mathbb{F}^k$ .*

(5) *Let  $a, b$  be blocks. We let  $a \cup b$  be undefined if  $\text{dom}(a) \cap \text{dom}(b) \neq \emptyset$  and  $a$  and  $b$  disagree on their intersection. Otherwise,  $a \cup b$  is defined as the union of the two functions.*

(6)  *$(\mathbb{F}^k, \cup)$  is a partial semigroup. The associative binary operation  $\cup$  lifts to  $\beta(\mathbb{F}^k)$ , the space of ultrafilters over  $\mathbb{F}^k$ , as follows (and we write  $\dot{\cup}$  for the lifted operation):*

$$\mathcal{U}_1 \dot{\cup} \mathcal{U}_2 = \{X \subseteq \mathbb{F}^k : \text{for } \mathcal{U}_1\text{-most } s, \text{ for } \mathcal{U}_2\text{-most } t, s \cup t \in X\}$$

*For details and history see [18, Section 4.1].*

(7) *If  $X$  is a subset of  $\mathbb{F}^k$ , we write  $\text{FU}(X)$  for the set of all finite unions of members of  $X$ . We write  $\text{FU}(\bar{a})$  instead of  $\text{FU}(\{a_n : n \in \omega\})$ .*

(8) *For  $X \subseteq \mathbb{F}^k$ , the set  $(\text{FU}(X))^\omega$  denotes the collection of all infinite unmeshed sequences in  $\text{FU}(X)$ .*

(9) *A filter over  $\mathbb{F}^k$  is a subset of  $\mathcal{P}(\mathbb{F}^k)$  that is closed under intersections and supersets and does not contain the empty set. The filter is called non-principal if it does not contain a finite set.*

The Milliken–Taylor ultrafilters  $\mathcal{U}_\alpha$  in the iterands  $\mathbb{M}(\mathcal{U}_\alpha)$  need to fulfil many tasks. The organisation of the tasks (in the proof of Theorem 2.12) is made possible by the following  $< \omega_1$ -complete preorder  $\sqsubseteq^*$ :



**Definition 2.2.** Given  $X$  and  $Y \subseteq \mathbb{F}^k$ , we say that  $Y$  is a condensation of  $X$  and we write  $Y \sqsubseteq X$  if  $Y \subseteq \text{FU}(X)$ . We say  $Y$  is almost a condensation of  $X$  and we write  $Y \sqsubseteq^* X$  iff there is a finite set  $Z$  such that  $Y \setminus Z$  is a condensation of  $X$ .

We use the definition mainly for  $Y = \text{range}(\bar{b})$  with  $\bar{b} \in (\mathbb{F}^k)^\omega$ ,  $X = \text{range}(\bar{a})$  for  $\bar{a} \in (\mathbb{F}^k)^\omega$ , and then we write  $\bar{b} \sqsubseteq \bar{a}$  for  $\text{range}(\bar{b}) \sqsubseteq \text{range}(\bar{a})$  and analogously for  $\sqsubseteq^*$ . We also call  $\bar{b}$  with  $\bar{b} \sqsubseteq \bar{a}$  a strengthening  $\bar{a}$  and we call  $\bar{b}$  with  $\bar{b} \sqsubseteq^* \bar{a}$  an almost strengthening  $\bar{a}$ . Strengthening can be imagined as the following procedure: First we drop (possibly infinitely many) blocks from  $\bar{a}$  such that infinitely many blocks remain. Then we merge finite groups of adjacent blocks to one block of  $\bar{b}$ .

**Definition 2.3.** A set  $\mathcal{C} \subseteq ((\mathbb{F}^k)^\omega)$  is called centred, if for any finite  $C \subseteq \mathcal{C}$  there is  $\bar{a} \in \mathcal{C}$  that is almost a condensation of any  $\bar{c} \in C$  and if  $\mathcal{C}$  is closed under finite alterations, i.e., if  $\bar{d} \in \mathcal{C}$  and  $\bar{e} =^* \bar{d}$  then  $\bar{e} \in \mathcal{C}$ .

We specialise  $\mathcal{C}$  further. For this, we recall some properties of filters over the set  $\mathbb{F}^k$ .

**Definition 2.4.** (1) A non-principal filter  $\mathcal{F}$  on  $\mathbb{F}^k$  is said to be an union filter if it has a basis of sets of the form  $\text{FU}(X)$  for  $X \subseteq \mathbb{F}^k$ .

(2) A non-principal filter  $\mathcal{F}$  on  $\mathbb{F}^k$  is said to be an ordered-union filter if it has a basis of sets of the form  $\text{FU}(\bar{d})$  for  $\bar{d} \in (\mathbb{F}^k)^\omega$ .

(3) Let  $\mu$  be an uncountable cardinal. An union filter is said to be  $< \mu$ -stable if, whenever it contains  $\text{FU}(X_\alpha)$  for  $X_\alpha \subseteq \mathbb{F}^k$ ,  $\alpha < \kappa$ , for some  $\kappa < \mu$ , then it also contains some  $\text{FU}(Y)$  for some  $Y$  such that for each  $\alpha$  there is  $n_\alpha$  with  $Y \restriction n_\alpha \in \text{FU}(X_\alpha)$  for  $\alpha < \kappa$ . For “ $< \omega_1$ -stable” we say “stable”.

(4) A stable ordered-union ultrafilter is also called a Milliken–Taylor ultrafilter.

(5) An ultrafilter is called idempotent if  $\mathcal{U} \dot{\cup} \mathcal{U} = \mathcal{U}$ .

Ordered-union ultrafilters need not exist, as their existence implies the existence of  $Q$ -points [6] and there are models without  $Q$ -points [29]. Even union ultrafilters need not exist, since according to [7, Theorem 38] the existence of a union ultrafilter implies the existence of at least two near-coherence classes of ultrafilters. Union ultrafilters are idempotent. Idempotent ultrafilters exist by the Ellis–Namakura Lemma [16, 32]. With the help of Hindman’s theorem one shows that under  $\text{MA}(\sigma\text{-centred})$  (even  $< 2^\omega$ -stable) Milliken–Taylor ultrafilters over  $\mathbb{F}^k$  exist [6]. We recall Hindman’s theorem:

**Theorem 2.5.** (Hindman, [17, Corollary 3.3]) If the set  $\mathbb{F}^k$  is partitioned into finitely many pieces then there is a set  $\bar{d} \in (\mathbb{F}^k)^\omega$  such that  $\text{FU}(\bar{d})$  is included in one piece.



The theorem also holds if instead of  $\mathbb{F}^k$  we partition only  $\text{FU}(\bar{c})$  for some  $\bar{c} \in (\mathbb{F}^k)^\omega$ , the homogeneous sequence  $\bar{d}$  given by the theorem is then a condensation of  $\bar{c}$ .

**Corollary 2.6.** *Under CH for every  $\bar{a} \in (\mathbb{F}^k)^\omega$  there is a Milliken–Taylor ultrafilter  $\mathcal{U}$  such that  $\text{FU}(\bar{a}) \in \mathcal{U}$ .*

We let for  $X \subseteq \mathbb{F}$ ,  $[X]_{<}^n$  be the set of increasing unmeshed  $n$ -sequences of members of  $X$ . For the evaluation of our forcings in Sections 3, 4, 5 we will frequently use Taylor’s theorem [37].

**Theorem 2.7.** *(Taylor [37].) Let  $\mathcal{U}$  be an Milliken–Taylor ultrafilter,  $n \in \omega$ . Let  $[\mathbb{F}^k]_{<}^n$  be partitioned into finitely many sets. Then there is  $A \in \mathcal{U}$  such that  $[\text{FU}(A)]_{<}^n$  is monochromatic.*

Now the rest of the section is devoted to the construction, under CH, of very particular Milliken–Taylor ultrafilters.

**Definition 2.8.** *Let  $\mathcal{U}$  be an ultrafilter over  $\mathbb{F}^k$  and  $i \in \{1, \dots, k\}$ . We define its  $i$ -fibre  $\mathcal{U}(i)$  by*

$$\mathcal{U}(i) = \{a^{-1}[\{i\}] : a \in \mathcal{U}\}.$$

If  $\mathcal{U}$  is an Milliken–Taylor ultrafilter over  $\mathbb{F}^k$  then  $\mathcal{U}(i)$  is a Milliken–Taylor ultrafilter with just the colour 1.

Ultrafilters over  $\mathbb{F}^k$  have some projections to filters over  $\omega$ :

**Definition 2.9.** *Let  $\mathcal{U}$  be a filter over  $\mathbb{F}^k$ .*

(1) *The core of  $\mathcal{U}$  is the filter  $\Phi(\mathcal{U})$  such that*

$$X \in \Phi(\mathcal{U}) \text{ iff } (\exists Y \in \mathcal{U})(\bigcup \{\text{dom}(a) : a \in Y\} \subseteq X).$$

(2) *Let  $i \in \{1, \dots, k\}$ . The core of  $\mathcal{U}(i)$  is the set  $\Phi(\mathcal{U}(i))$  such that*

$$X \in \Phi(\mathcal{U}(i)) \text{ iff } (\exists Y \in \mathcal{U})(\bigcup \{a^{-1}[\{i\}] : a \in Y\} \subseteq X).$$

(3) *Let  $s \in \mathbb{F}^k$ . We write*

$$\min_k(s) = (\min(s^{-1}[\{i\}]) : i = 1, \dots, k) \in \omega^k.$$

$$\max_k(s) = (\max(s^{-1}[\{i\}]) : i = 1, \dots, k) \in \omega^k.$$

(4) *The minimum projection of  $\mathcal{U}$  is the set*

$$\hat{\min}_k(\mathcal{U}) = \{\min_k[Y] : Y \in \mathcal{U}\},$$

where

$$\min_k[Y] = \{\min_k(y) : y \in Y\},$$

and analogously we define the maximum projection  $\hat{\max}_k(\mathcal{U})$ .

(5) Let  $i \in \{1, \dots, k\}$ . The minimum projection of  $\mathcal{U}(i)$  is the set

$$\hat{\min}(\mathcal{U}(i)) = \{\min[Y(i)] : Y \in \mathcal{U}\},$$

where

$$\min[Y(i)] = \{(\min_k(y))_i : y \in Y\} = \{\min(y^{-1}[\{i\}]) : y \in Y\}.$$

Analogously we define the maximum projection  $\hat{\max}(\mathcal{U}(i))$ .

(6) For  $Y = \{a_n : n \in \omega\}$  with  $\bar{a} \in (\mathbb{F}^k)^\omega$  we write  $\min_k[\bar{a}]$  for  $\min_k[Y]$  and  $\min[\bar{a}(i)]$  for  $\min[Y(i)]$ .

The cores are filters over  $\omega$ . The projections are filters over  $\omega^k$  and over  $\omega$  respectively, the projections of ultrafilters are ultrafilters.

**Definition 2.10.** Let  $\mathcal{H} \subseteq [\omega]^\omega$ .  $\mathcal{H}$  is called *ultra by finite-to-one* iff there is a finite-to-one function  $f$  such that  $f(\mathcal{H})$  is an ultrafilter.

This notion is mainly used for filters and semifilters but can also be used for arbitrary subsets of  $[\omega]^\omega$ . In Observation 3.9 we will see that a coideal is ultra by finite-to-one iff its dual filter is so.

If  $\mathcal{U}$  is ultra over  $\mathbb{F}$ , then  $\Phi(\mathcal{U})$  is not diagonalised (see [15, Prop. 2.3]) and also all finite-to-one images of  $\Phi(\mathcal{U})$  are not diagonalised (same proof). Hence, by Talagrand [36]  $\Phi(\mathcal{U})$  is not meagre.  $\Phi(\mathcal{U})$ , though, is not ultra by finite-to-one by [7, Theorem 3.8], nor is one of the  $\Phi(\mathcal{U}(i))$ .

**Definition 2.11.** The Rudin–Blass ordering for filters over  $\omega$  is defined as follows: Let  $\mathcal{F} \leq_{RB} \mathcal{G}$  iff there is a finite-to-one  $f$  such that  $f(\mathcal{F}) \subseteq f(\mathcal{G})$ .

Now we rework and extend Blass' [6]. There will be many ultrafilters in this section. We adopt the following conventions:  $\mathcal{R}, \mathcal{S}, \mathcal{W}, \mathcal{E}$  (possibly with indices) stand for ultrafilters over  $\omega$ .  $\mathcal{R}, \mathcal{S}$  are usually selective,  $\mathcal{E}$  is a  $P$ -point.  $\mathcal{V}$  is mainly used for ultrafilters over  $\omega^k$ . The letter  $\mathcal{U}$  is used for ultrafilters over  $\mathbb{F}^k$ , often Milliken–Taylor ultrafilters.

Now we can state the main result of this section:

**Theorem 2.12.** Given selective ultrafilters  $\mathcal{R}_1, \dots, \mathcal{R}_k$  and  $\mathcal{S}_1, \dots, \mathcal{S}_k$  such that each of the  $\mathcal{R}_j$  is not nearly coherent to any of the  $\mathcal{S}_i$  and given a ultrafilters  $\mathcal{E}, \mathcal{W}$  that are not nearly coherent to any of the  $\mathcal{R}_i, \mathcal{S}_j$ , we can find an Milliken–Taylor ultrafilter  $\mathcal{U}$  over  $\mathbb{F}^k$  such that

$$\begin{aligned} \text{For } i = 1, \dots, k, \hat{\min}(\mathcal{U}(i)) &= \mathcal{R}_i, \\ \text{for } i = 1, \dots, k, \hat{\max}(\mathcal{U}(i)) &= \mathcal{S}_i, \\ \Phi(\mathcal{U}) &\not\leq_{RB} \mathcal{E}, \mathcal{W}. \end{aligned} \tag{2.1}$$

The proof will take the rest of this section. We note that for the proof of the main theorem only the following weaker variant for Theorem 2.12 is used:

**Theorem 2.13.** *Given pairwise non-nearly coherent selective ultrafilters  $\mathcal{R}_1, \dots, \mathcal{R}_k$  and  $\mathcal{S}_1, \dots, \mathcal{S}_k$  and given ultrafilters  $\mathcal{E}, \mathcal{W}$  that are not nearly coherent to any of the  $\mathcal{R}_i, \mathcal{S}_j$ , we can find an Milliken–Taylor ultrafilter  $\mathcal{U}$  over  $\mathbb{F}^k$  such that*

$$\begin{aligned} \text{For } i = 1, \dots, k, \hat{\min}(\mathcal{U}(i)) &= \mathcal{R}_i, \\ \text{for } i = 1, \dots, k, \hat{\max}(\mathcal{U}(i)) &= \mathcal{S}_i, \\ \Phi(\mathcal{U}) &\not\leq_{\text{RB}} \mathcal{E}, \mathcal{W}. \end{aligned}$$

A reader only interested in the proof of the main theorem can skip some parts of the proof of Theorem 2.12 so that a proof of Theorem 2.13 remains. We indicate the parts that can be skipped. The reason for Theorem 2.12 is curiosity: We just wanted to investigate the optimal premise.

First we define  $\otimes$ :

**Definition 2.14.** *Let for  $i = 1, \dots, k+1$ ,  $\mathcal{R}_i$  be ultrafilter over  $\omega$ .*

$$\begin{aligned} (1) \quad \mathcal{R}_1 \otimes \mathcal{R}_2 &= \left\{ X \subseteq \omega^2 : \{m : \{n : (m, n) \in X\} \in \mathcal{R}_2\} \in \mathcal{R}_1 \right\}. \\ (2) \quad \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \dots \otimes \mathcal{R}_{k+1} &= \left\{ X \subseteq \omega^{k+1} : \right. \\ &\quad \left. \{\vec{m} \in \omega^k : \{n : (\vec{m}, n) \in X\} \in \mathcal{R}_{k+1}\} \in \mathcal{R}_1 \otimes \dots \otimes \mathcal{R}_k \right\}. \end{aligned}$$

We simplify the situation by considering the sharper conclusion

$$\begin{aligned} (2.2) \quad \hat{\min}_k(\mathcal{U}) &= \mathcal{R}_1 \otimes \dots \otimes \mathcal{R}_k, \\ \hat{\max}_k(\mathcal{U}) &= \mathcal{S}_1 \otimes \dots \otimes \mathcal{S}_k, \\ \Phi(\mathcal{U}) &\not\leq_{\text{RB}} \mathcal{E}, \mathcal{W}. \end{aligned}$$

This transition to the sharper conclusion can be seen as destruction of the symmetry of the “colours”  $\{1, \dots, k\}$ : From now on we will look for Milliken–Taylor ultrafilters  $\mathcal{U}$  over  $\mathbb{F}^k$  that concentrate on

$$\begin{aligned} (2.3) \quad \{a \in \mathbb{F}^k : \min(a^{-1}[\{1\}]) &< \dots < \min(a^{-1}[\{k\}]) \\ &< \max(a^{-1}[\{1\}]) < \dots < \max(a^{-1}[\{k\}])\} \end{aligned}$$

The focus on blocks in this set will help to organise the construction of the iterands. By Hindman’s theorem, for any Milliken–Taylor ultrafilter  $\mathcal{U}$  over  $\mathbb{F}^k$  there are two linear orders  $<_{o, \min}$  and  $<_{o, \max}$  on  $k$  such that  $\mathcal{U}$  concentrates on

$$\begin{aligned} \{a \in \mathbb{F}^k : \{\min(a^{-1}[\{1\}]), \dots, \min(a^{-1}[\{k\}])\} \\ \text{is ordered according to } <_{o, \min} \text{ and lies before} \\ \{\max(a^{-1}[\{1\}]), \dots, \max(a^{-1}[\{k\}])\} \\ \text{that is ordered according to } <_{o, \max}\} \end{aligned}$$

**Definition 2.15.** (1) *Let  $k \in \omega \setminus \{0\}$ ,  $\vec{x} \in \omega^k$ ,  $i \in \{1, \dots, k\}$ , we let  $\text{pr}_i(\vec{x}) = x_i$ . For  $X \subseteq \omega^k$  and  $i \in \{1, \dots, k\}$ , we let  $\text{pr}_i[X] = \{\text{pr}_i(\vec{x}) : \vec{x} \in X\} = \{x_i : \exists x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k \vec{x} \in X\}$ .*

- (2) Let  $k \in \omega \setminus \{0\}$ ,  $\mathcal{V}$  an ultrafilter over  $\omega^k$ . Let  $i \in \{1, \dots, k\}$ . Again for the double lifting we use the round brackets again and define:  $\text{pr}_i(\mathcal{V}) = \{X \subseteq \omega : \exists Y \in \mathcal{V} \text{ pr}_i[Y] \subseteq X\}$ .
- (3)  $k \in \omega \setminus \{0\}$ .  $\Delta_k = \{\vec{a} \in \omega^k : a_1 < \dots < a_k\}$ .
- (4)  $k \in \omega \setminus \{0\}$ . We define the upper triangle in  $\omega^{2k}$  as  $\text{uppertriangle}^{2k} = \{(\vec{m}, \vec{n}) \in \omega^{2k} : \max(\vec{m}) < \min(\vec{n})\}$ .

A reader who is only interested in the proof of the main theorem can skip the following discussion and go ahead to Definition 2.18. We discuss in what respect Equation (2.2) is a stronger requirement than Equation (2.1):

The first issue is: Let  $\mathcal{R}_i$  be an ultrafilter for  $i = 1, \dots, k$ . How many ultrafilters  $\mathcal{V}$  over  $\omega^k$  are there with  $\text{pr}_i(\mathcal{V}) = \mathcal{R}_i$ ? Let  $k = 2$ : If  $\mathcal{R}_1$  is not nearly coherent to  $\mathcal{R}_2$  and  $\mathcal{R}_1, \mathcal{R}_2$  are non-nearly coherent  $P$ -points then there is  $\mathcal{R}_1 \otimes \mathcal{R}_2$  and  $\{\text{sw}[X] : X \in \mathcal{R}_2 \otimes \mathcal{R}_1\}$  where  $\text{sw}[X] = \{(y, x) : (x, y) \in X\}$  and these two are the only ones, as is remarked in [6, Page 97], which is based on [33]. Nearly coherent selective ultrafilters are isomorphic, that means: There are  $A_i \in \mathcal{R}_i$  and a bijection  $f: A_1 \rightarrow A_2$  such that  $\mathcal{R}_2$  is generated by  $\{f[X] : X \in \mathcal{R}_1\}$  see [4]. We call  $f$  an isomorphism from  $\mathcal{R}_1$  to  $\mathcal{R}_2$ . If  $f$  is such an isomorphism then in addition to  $\mathcal{R}_1 \otimes \mathcal{R}_2$  and  $\{\text{sw}[X] : X \in \mathcal{R}_2 \otimes \mathcal{R}_1\}$  also  $\{(x, f(x)) : x \in X\} : X \in \mathcal{R}_1\}$  is an ultrafilter over  $\omega^2$  with projections  $\mathcal{R}_i$ . If the  $\mathcal{R}_i$  are selective there three are the only ones [6, page 97]. Having opted for the order of the colours as in (2.3) we will from now on always work with  $\mathcal{R}_1 \otimes \mathcal{R}_2$  and disregard the other possibilities.

If the projections are not  $P$ -points, then the classification of the the ultrafilter over  $\omega^k$  with given projections becomes a complex topic, see for example [9] for recent work on this topic. However, when heading for Milliken–Taylor ultrafilters we necessarily have selective projections, see below.

Now we show that under the side condition that the premises of Theorem 2.12 are necessary.

The requirement on non-near coherence in Theorem 2.12 is necessary: The fact that each  $\mathcal{R}_i$  is not nearly coherent to  $\mathcal{S}_j$  follows from the following slight generalisation of [7, Theorem 38].

**Theorem 2.16.** *If  $\mathcal{U}$  is a union ultrafilter over  $\mathbb{F}^k$  and  $(i, j) \in \{1, \dots, k\}^2$  then  $\hat{\text{m}}\text{ax}(\mathcal{U}(i))$  and  $\hat{\text{m}}\text{in}(\mathcal{U}(j))$  are non-nearly-coherent  $P$ -points.*

*Proof.* The  $P$ -point part is proved in [10]. We assume that  $f$  is finite-to-one and  $f(\hat{\text{m}}\text{ax}(\mathcal{U}(i))) = f(\hat{\text{m}}\text{in}(\mathcal{U}(j)))$  for a pair  $(i, j) \in \{1, \dots, k\}^2$ . Moreover we assume that  $f$  is so flat that in case  $\hat{\text{m}}\text{ax}(\mathcal{U}(i))$  and  $\hat{\text{m}}\text{ax}(\mathcal{U}(j))$  are nearly coherent  $f(\hat{\text{m}}\text{ax}(\mathcal{U}(i))) = f(\hat{\text{m}}\text{ax}(\mathcal{U}(j)))$  or in case  $\hat{\text{m}}\text{in}(\mathcal{U}(i))$  and  $\hat{\text{m}}\text{in}(\mathcal{U}(j))$  are nearly coherent then  $f(\hat{\text{m}}\text{in}(\mathcal{U}(i))) = f(\hat{\text{m}}\text{in}(\mathcal{U}(j)))$ . W.l.o.g. we assume

that  $f$  is non-decreasing and surjective. We let  $I_n = f^{-1}[\{n\}]$ . We let  $E$  be

$$E = \{s \in \mathbb{F}^k : |\{r \in \omega : I_r \cap s^{-1}[\{i\}] \neq \emptyset\}| \text{ is even and} \\ |\{r \in \omega : I_r \cap s^{-1}[\{j\}] \neq \emptyset\}| \text{ is even and} \\ |\{r \in \omega : I_r \cap (s^{-1}[\{i, j\}]) \neq \emptyset\}| \text{ is even.}\}$$

Since  $\mathcal{U}$  is an ultrafilter over  $\mathbb{F}^k$ , we have  $E \in \mathcal{U}$  or  $\mathbb{F}^k \setminus E \in \mathcal{U}$ . Since  $\mathcal{U}$  is a union ultrafilter,  $E \in \mathcal{U}$ , and there is  $A \subseteq \mathbb{F}^k$  such that  $\text{FU}(A) \subseteq E$ . We shrink  $A$  in case  $\hat{\text{max}}(\mathcal{U}(i))$  and  $\hat{\text{max}}(\mathcal{U}(j))$  are not nearly coherent such that  $f[\max[A(i)]] \cap f[\max[B(j)]] = \emptyset$  for some  $B \in \mathcal{U}$  or in case  $\hat{\text{min}}(\mathcal{U}(i))$  and  $\hat{\text{min}}(\mathcal{U}(j))$  are not nearly coherent such that  $f[\min[A(i)]] \cap f[\min[C(j)]] = \emptyset$  for some  $C \in \mathcal{U}$  or both; thereafter we replace  $A$  by  $A \cap B \cap C$ .

Since we assumed near coherence we have

$$f[\max[A(i)]] \cap f[\min[A(j)]] \neq \emptyset.$$

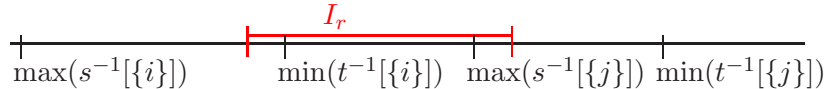
So there is an interval  $I_r$  that meets  $\max[A(i)]$  and  $\min[A(j)]$ .

First case:  $\hat{\text{max}}(\mathcal{U}(i))$  and  $\hat{\text{max}}(\mathcal{U}(j))$  are nearly coherent. Since the  $I_r$  form an interval partition, in the same  $I_r$  there is, by our premise on the flatness of  $f$ , also a point from  $\hat{\text{max}}(\mathcal{U}(j))$ . Now we argue as in Blass' proof for  $\mathcal{U}(j)$  and  $\max_j$  and  $\min_j$ : Let  $s, t \in A$  with  $\max(s^{-1}[\{j\}])$  and  $\min(t^{-1}[\{j\}])$  both in  $I_r$ .  $\{r' : I_{r'} \cap (s \cup t)^{-1}[\{j\}] \neq \emptyset\}$  is the union of  $\{r' : I_{r'} \cap s^{-1}[\{j\}] \neq \emptyset\}$  and  $\{r' : I_{r'} \cap t^{-1}[\{j\}] \neq \emptyset\}$  and the latter two intersect just in  $\{r\}$ . This contradicts  $s \cup t \in A \subseteq E$  and the second clause in the definition of  $E$ .

Second case:  $\hat{\text{min}}(\mathcal{U}(i))$  and  $\hat{\text{min}}(\mathcal{U}(j))$  are nearly coherent. Since the  $I_r$  form an interval partition, in the same  $I_r$  there is, by our premise on the flatness of  $f$ , also a point from  $\hat{\text{min}}(\mathcal{U}(i))$ . Now we argue as in Blass' proof for  $\mathcal{U}(i)$  and  $\max[A(i)]$  and  $\min[A(i)]$  and we use the first clause of the definition of  $E$ .

Third case: Neither  $\hat{\text{max}}(\mathcal{U}(i))$  and  $\hat{\text{max}}(\mathcal{U}(j))$  nor  $\hat{\text{min}}(\mathcal{U}(i))$  and  $\hat{\text{min}}(\mathcal{U}(j))$  are nearly coherent. We assume  $i \leq j$ , the other case is the mirrored situation (since here we do not yet work with Equation (2.3) the other case appears). The constellation  $\max(\text{dom}(t)) < \max(\text{dom}(s))$  is excluded since it contradicts our choice of  $A$  that implies  $\max(t^{-1}[\{j\}]) \notin I_r$ . The case  $s = t$  does not occur since  $s = t$  implies  $\max(s^{-1}[\{i\}]) = \max(t^{-1}[\{i\}])$ , so by our choice of  $f$ ,  $\min(t^{-1}[\{i\}])$  cannot lie in  $I_r$ , in contradiction to our assumption. So only the case  $\max(\text{dom}(s)) < \max(\text{dom}(t))$  needs to be considered.

Now, since none of  $\max[A(i)]$  meets  $I_r$  and none of  $\min[A(j)]$  meets  $I_r$  we have  $\{r' : I_{r'} \cap (s \cup t)^{-1}[\{i, j\}] \neq \emptyset\}$  is the union of  $\{r' : I_{r'} \cap s^{-1}[\{i, j\}] \neq \emptyset\}$  and  $\{r' : I_{r'} \cap t^{-1}[\{i, j\}] \neq \emptyset\}$  and the latter two intersect just in  $\{r\}$ .



Now the third clause in the definition of  $E$  contradicts the fact that  $s \cup t \in \text{FU}(A) \subseteq E$ . We drew a sketch for a meshed situation. The theorem used just the union-ultrafilter property. Ordered-union is not needed.  $\square$

We demand in the premises of Theorem 2.12 selectivity for the  $\mathcal{R}_i$ ,  $\mathcal{S}_j$  though in the light of Theorem 2.19 below this seems to be unnaturally much. However, for *ordered-union* ultrafilters  $\mathcal{U}$  over  $\mathbb{F}^k$ ,  $Q$ -points appear as minimum and maximum projections:

**Proposition 2.17.** [6, Prop. 3.9] *If  $\mathcal{U}$  is a ordered-union ultrafilter over  $\mathbb{F}^k$ , then  $\hat{\min}(\mathcal{U}(i))$  and  $\hat{\max}(\mathcal{U}(i))$  are  $Q$ -points.*  $\square$

Thus we showed the necessity of the premises in Theorem 2.12, and we now turn to the proof of Theorem 2.12:

**Definition 2.18.** ([7]) *A finite sequence  $\langle \mathcal{R}_1, \dots, \mathcal{R}_k \rangle$  of ultrafilters is homogenic if, for every partition of  $[\omega]^k$  into two pieces, there are sets  $A(\mathcal{R}_i)$ , one for each ultrafilter  $\mathcal{R}_i$  in the given sequence, such that one piece of the given partition contains all of the sets  $\{a_1 < \dots < a_k\}$  with  $a_i \in A(\mathcal{R}_i)$  for all  $i$ .*

We use this definition only for finitely many ultrafilters. Then the fine point, that  $A(\mathcal{R}_i)$  depends only on  $\mathcal{R}_i$  and not on  $i$  does not play a role.

**Theorem 2.19.** [7, Theorem 19, (1)(2)(5)]. *Let  $\mathcal{W}_1$  and  $\mathcal{W}_2$  be two different ultrafilters (non-principal and over  $\omega$  as always). The following are equivalent.*

- (1)  $\langle \mathcal{W}_1, \mathcal{W}_2 \rangle$  is homogenic.
- (2) The sets  $\{A \times \omega : A \in \mathcal{W}_1\} \cup \{\omega \times B : B \in \mathcal{W}_2\} \cup \{\Delta_2\}$  generate an ultrafilter (namely  $\mathcal{W}_1 \otimes \mathcal{W}_2$ ).
- (3)  $\mathcal{W}_2$  is a  $P$ -point, and, for any finite-to-one functions  $f, g: \omega \rightarrow \omega$ , there exist sets  $A \in \mathcal{W}_1$  and  $B \in \mathcal{W}_2$  with the following property. Whenever  $a \in A$  and  $b \in B$  and  $a < b$ , then  $f(a) \neq g(b)$ .

**Theorem 2.20.** [7, Theorem 26] *For any finite sequence  $\langle \mathcal{W}_1, \dots, \mathcal{W}_k \rangle$  of non-principal pairwise different ultrafilters over  $\omega$ , the following are equivalent.*

- (1)  $\langle \mathcal{W}_1, \dots, \mathcal{W}_k \rangle$  is homogenic.
- (2) The sets  $\{\omega^{i-1} \times B \times \omega^{k-i-1} : B \in \mathcal{W}_i\}$ ,  $i = 1, \dots, k$ , together with  $\Delta_k$  generate an ultrafilter over  $\omega^k$  (namely  $\mathcal{W}_1 \otimes \dots \otimes \mathcal{W}_k$ ).
- (3) Each pair of consecutive ultrafilters  $\langle \mathcal{W}_i, \mathcal{W}_{i+1} \rangle$  (for  $1 \leq i < k$ ) is homogenic.

We shall use the equivalence of (2) and (3) in our applications.

The following lemma is used for Theorem 2.12 but not for Theorem 2.13.

**Lemma 2.21.** *If  $\mathcal{R}_1 \neq \mathcal{R}_2$  are selective and nearly coherent ultrafilters, then  $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$  is homogenic.*

*Proof.* We show (2) of Theorem 2.19.  $X \in \mathcal{R}_1 \otimes \mathcal{R}_2$ . Fix  $A \in \mathcal{R}_1$  such that  $A = \{x \in \omega : \{y : (x, y) \in X\} \in \mathcal{R}_2\}$ . Let  $B_x = \{y : (x, y) \in X\}$ . Then we have  $B_x \in \mathcal{R}_2$  for  $x \in A$ . W.l.o.g. let the  $B_x$  be decreasing, that is for  $x_1 < x_2 \in A$ ,  $B_{x_1} \supseteq B_{x_2}$ . Nearly coherent selective ultrafilters are isomorphic [4]. Let  $f: A' \rightarrow B'$  be bijective and an isomorphism from  $\mathcal{R}_1$  to  $\mathcal{R}_2$ , and  $A' \in \mathcal{R}_1$ ,  $B' \in \mathcal{R}_2$ ,  $A' \subseteq A$ . Take  $\Pi = \langle I_n : n \in \omega \rangle$  an interval partition of  $\omega$  such that  $\forall n \forall i \in I_n f(i) \in I_{n-1} \cup I_n \cup I_{n+1}$  (see [5, Lemma 2.1]). Take  $A'' \in \mathcal{R}_1$ ,  $A'' \subseteq A'$  such that for any  $n, j = 0, 1$ ,  $A'' \cap I_{2n-1+j} \cup I_{2n+j}$  has at most one element. Then we have between any two intervals met by  $A''$  there is one interval that is not met by  $A''$ . Let for  $y \in I_n$ ,  $g(y) = \max(I_{n'}) + 1$  where  $n'$  is the next index  $> n + 1$  such that  $I_{n'} \cap A'' \neq \emptyset$ . Since  $\mathcal{R}_2$  is Ramsey there is  $B \in \mathcal{R}_2$ ,  $B \subseteq f[A'']$ , such that  $\forall y \in B B \setminus (y + 1) \subseteq \bigcap_{i \leq g(y)} B_i$ . We show:  $A'' \times B \cap \Delta_2 \subseteq X$ . Let  $x \in A''$ ,  $y > x$ ,  $y \in B$ , we have to show that  $(x, y) \in X$ : We take  $n$  such that  $x \in I_n$ . If  $y = f(x) > x$ , then  $B \setminus (y + 1) \subseteq B_y \subseteq B_x$  as desired. If  $y = f(x') > x$ , for  $x' > x$  then  $B \setminus (y + 1) \subseteq B_{g(x')}$ . Since  $g(x') \geq y$ ,  $B \subseteq (y + 1) \subseteq B_x$ . So in both cases we have  $(x, y) \in X$ . This finishes the proof of  $A'' \times B \cap \Delta_2 \subseteq X$ . Now we have property (2) of Theorem 2.19 and hence  $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$  is homogenic.  $\square$

We put the lemma together with Theorem 2.20. For 2.13 one could add the condition that the  $\mathcal{R}_i$  are pairwise non-nearly coherent.

**Corollary 2.22.** *If  $\mathcal{R}_i$ ,  $i = 1, \dots, k$ , are selective and pairwise different then  $\{\omega^{i-1} \times B \times \omega^{k-i} : B \in \mathcal{R}_i\}$ ,  $i = 1, \dots, k$ , together with*

$$\{(a_1, \dots, a_k) \in \omega^k : a_1 < \dots < a_k\}$$

*generates  $\bigotimes \mathcal{R}_i$ .*

*Proof.* If  $\mathcal{R}_i, \mathcal{R}_{i+1}$  are selective and nearly coherent, then it is the previous lemma. If they are selective and not nearly coherent, then item (3) of Theorem 2.19 is true. In any case, all the pairs  $\langle \mathcal{R}_i, \mathcal{R}_{i+1} \rangle$ ,  $i = 1, \dots, k - 1$  are homogenic. Now the conclusion follows from the implication from (3) to (2) Theorem 2.20.  $\square$

The following lemma is well-known for  $k = 2$ , see, e.g. [14, Prop. 2.2]. We write  $\Pi = \langle I_n : n \in \omega \rangle$  for partitions of  $\omega$  into intervals such that  $\bigcup I_n = \omega$  and  $I_n < I_{n+1}$ .

**Lemma 2.23.** *(The buffer lemma)*

- (1) *Let  $\mathcal{R}_i$ ,  $i = 1, \dots, k$ , be pairwise non nearly coherent ultrafilters over  $\omega$ . Let  $\Pi = \langle I_n : n < \omega \rangle$  be an interval partition. Then there is  $X_i \in \mathcal{R}_i$  such that for  $i \neq j$ ,  $X_i, X_j$  do not meet the same interval nor adjacent intervals.*



(2) Let  $\mathcal{R}_i$ ,  $i = 1, \dots, k$ , be pairwise non nearly coherent  $Q$ -points. Let  $\Pi = \langle I_n : n < \omega \rangle$  be an interval partition. Then there is  $X_i^0 \in \mathcal{R}_i$  such that  $X_i^0$  meets each  $\Pi$  interval at most once and such that for  $i \neq j$ ,  $X_i^0, X_j^0$  do not meet the same interval nor adjacent intervals.

*Proof.* (1) We let  $f_{2\Pi}(x) = n$  for  $x \in I_{2n} \cup I_{2n+1}$  for  $x, n \in \omega$ , and  $f_{2\Pi+1}(x) = n$  for  $x \in I_{2n+1} \cup I_{2n+2}$  for  $n \in \omega$  and  $f_{2\Pi+1}(x) = 0$  for  $x \in I_0 \cup I_1 \cup I_2$ . Then we take (in  $k-1+k-2+\dots+1 = k(k-1)/2$  substeps) sets  $X_i \in \mathcal{R}_i$  such that for  $p = 0, 1$ ,  $i \neq j \in \{1, \dots, k\}$ ,  $f_{2\Pi+p}[X_i] \cap f_{2\Pi+p}[X_j] = \emptyset$ . So it  $X_i$  and  $X_j$  meet two adjacent intervals the first interval must have an even index by the equation for  $p = 0$  and an odd index by the equation for  $p = 1$ , so they cannot meet at all two adjacent intervals. (2) We take  $X_i$  as in (1). Then we take  $X_i^0 \in \mathcal{R}_i$ ,  $X_i^0 \subseteq X_i$  that meet each interval at most once. This is possible since the  $\mathcal{R}_i$  are  $Q$ -points.  $\square$

The next lemma is for one group of pairwise different, paves nearly coherent selective ultrafilters. It generalises the proof of Lemma 2.21. Again it is not needed for the proof of Theorem 2.13.

**Lemma 2.24.** *Let  $\mathcal{R}_i$  be selective and nearly coherent. Let  $A_n \in \mathcal{R}_1 \otimes \dots \otimes \mathcal{R}_k$ ,  $n \in \omega$ , be decreasing. Then there is  $X \in \mathcal{R}_1 \otimes \dots \otimes \mathcal{R}_k$  such that*

$$\forall \vec{n} \in X \ X \setminus (\max(\vec{n}) + 1)^k \subseteq A_{\max(\vec{n})}.$$

*Proof.* By the previous lemmata we may assume that there are  $A_{i,n} \in \mathcal{R}_i$  such that  $\prod_{i=1}^k A_{i,n} \cap \Delta_k \subseteq A_n$  and such that each sequence  $A_{i,n}$  is descending. We use that nearly coherent selective ultrafilters are isomorphic [4] and take (again in  $k(k-1)/2$  substeps) bijections  $f_{i,j} : X_i \rightarrow X_j$  so that  $f_{i,j}$  is an isomorphism from  $\mathcal{R}_i$  to  $\mathcal{R}_j$  and  $X_i \in \mathcal{R}_i$  is the same for  $j$  and such that  $f_{j,1} \circ f_{1,i} = f_{j,i}$ . We take an interval partition  $\Pi = \langle I_n : n \in \omega \rangle$  such that for any  $i \neq j$ ,  $x \in I_n$ ,  $f_{i,j}(x) \in I_{n-1} \cup I_n \cup I_{n+1}$ . Then we take  $X_1^1 \in \mathcal{R}_1$  such that  $X_1^1 \subseteq \bigcap_{i=1}^k f_{1,i}[A_{i,0} \cap X_i]$  and such that  $X_1^1$  meets each interval at most once and does not meet two adjacent intervals nor any two intervals that have just one or two interval in between. We let  $X_i^1 = f_{1,i}[X_1^1]$ . Thus we have neatly separated clusters of at most three intervals containing  $f_{i,j}(x)$ ,  $i, j \in \{1, \dots, k\}$ , then a gap of at least one free interval, then the next cluster  $f_{i,j}(y)$ ,  $i \neq j \in \{1, \dots, k\}$  such that the union of these clusters equals  $\bigcup_{i=1}^k X_i^1$ . Again we define  $g$  by letting for  $x \leq \max(I_n)$ ,  $n' \geq n+2$  being minimal such that  $I_{n'} \cap X_1^1 \neq \emptyset$  and setting  $g(x) = \max(I_{n'+1})$ . By selectivity we have  $X_i^2 \subseteq X_i^1$  such that

$$\forall x \in X_i^2 \ X_i^2 \setminus (x+1) \subseteq A_{i,g(x)}$$

We thin out the  $X_i^2$  to  $X_i^3 \in \mathcal{R}_i$  such that with  $f_{1,i}[X_1^3] = X_i^3$ . Now we let  $X = \prod_i X_i^3 \cap \Delta_k$ . For any  $(x_1, x_2, \dots, x_k) \in X$  for any  $(y_1, y_2, \dots, y_k) \in X$  with  $y_1 > x_k$  we have  $y_i \in X_i^3 \setminus (x_i+1) \subseteq A_{i,g(x_i)} \subseteq A_{i,x_k}$ . So we have for every  $\vec{x} \in X = (\prod_{i=1}^k X_i^2 \cap \Delta_k)$ ,  $X \setminus (x_k+1)^k \subseteq A_{x_k}$  as desired.  $\square$

Now selectivity is used to get the alternating order  $(\min_k(s), \max_k(s)) = (\vec{x}, \vec{y})$  with  $x_1 < x_2 < \dots < x_k < y_1 < \dots < y_k$  for  $\mathcal{U}$ -many  $s \in \mathbb{F}^k$  for a suitably ultrafilter  $\mathcal{U}$  over  $\mathbb{F}^k$ .

We identify  $[\omega]^k$  with the set of increasing functions  $f: k \rightarrow \omega$  and we write them as  $\vec{x} = (f(1), \dots, f(k))$  and sometimes we write only their range. We write  $\vec{x} < \vec{y}$  for  $\max \vec{x} < \min \vec{y}$ . The next lemma is for  $r = 2$ ,  $i_1 = 1$ ,  $i_2 = 1$  in Blass [6, 1.2]. The reader who wants to focus on Theorem 2.13 can let  $i_\varrho = 1$  for  $\varrho = 1, \dots, r$ .

**Lemma 2.25.** *(Strong generalisation of [6, 1.2]) If  $\mathcal{R}_i$ ,  $i = 1, \dots, k$ , are pairwise different selective ultrafilters and  $k = i_1 + \dots + i_r$  and for any  $\varrho = 1, \dots, r$ ,  $\mathcal{R}_i$  is nearly coherent to  $\mathcal{R}_j$  for  $i, j \in [\sum_{\sigma < \varrho} i_\sigma + 1, \sum_{\sigma \leq \varrho} i_\sigma]_{\mathbb{N}} =: K_\varrho$  via  $f_{i,j}$ . and for  $1 \leq \varrho \leq r$  if  $i, j \in 1, \dots, k$ , are not in the same  $\varrho$  interval, then they are not nearly coherent. Let  $A_{i,n} \in \mathcal{R}_i$ ,  $n \in \omega$ , be descending.  $A_n = \prod_{\varrho=1}^r \prod_{i \in K_\varrho} A_{i,n} \cap \Delta_k$ . Then for  $\varrho = 1, \dots, r$  there is  $X_\varrho \in \bigotimes_{i \in K_\varrho} \mathcal{R}_i$  such that*

- (1) for any  $\varrho$ ,  $\forall \vec{n} \in X_\varrho X_\varrho \setminus (\max(\vec{n}) + 1)^{i_\varrho} \subseteq \prod_{i \in K_\varrho} A_{i, \max(\vec{n})}$ , and
- (2) for any  $\varrho = 1, r-1$   $\forall \vec{x} \in \prod_{\sigma \leq \varrho} X_\sigma, \vec{y} \in \prod_{\sigma > \varrho} X_\sigma (\max(\vec{x}) < \min(\vec{y}) \rightarrow \vec{y} \in \prod_{\sigma > \varrho} \prod_{i \in K_\sigma} A_{i, \max \vec{x}} = A_{\vec{x}})$ , and
- (3) for any  $\varrho = 1, r-1$   $\forall \vec{x} \in \prod_{\sigma \leq \varrho} X_\sigma, \vec{y} \in \prod_{\sigma > \varrho} X_\sigma (\max(\vec{y}) < \min(\vec{x}) \rightarrow \vec{x} \in \prod_{\sigma \leq \varrho} \prod_{i \in K_\sigma} A_{i, \max \vec{y}} = B_{\vec{y}})$ , and
- (4) the elements of  $X_1, X_2, \dots, X_r$  alternate in a strong sense: there is an enumeration  $\langle \vec{z}_i : i < \omega \rangle$  of  $X_1 \cup \dots \cup X_r$  such that for any  $n$ ,  $\varrho = 1, \dots, r$ ,  $z_{rn+\varrho} \in X_\varrho$  and  $\max(z_{rn+\varrho}) < \min(z_{r(n+1)+\varrho})$ . The  $z_{rn+\varrho}$  are increasing vectors by the definition of  $\bigotimes_{i \in K_\varrho} \mathcal{R}_i$ .

*Proof.* The proof is by induction on  $r$ . The case  $r = 1$  is Lemma 2.24. Now we let  $r \geq 2$  and we carry out the induction step from  $r-1$  to  $r$ . This induction step is as in [6, 1.2]. We let

$$\prod_{\sigma > \varrho} \prod_{i \in K_\sigma} A_{i, \max \vec{x}} = A_{\vec{x}}$$

and

$$\prod_{\sigma \leq \varrho} \prod_{i \in K_\sigma} A_{i, \max \vec{y}} = B_{\vec{y}}.$$

We may assume that the  $A_{\vec{y}}$  and the  $B_{\vec{x}}$  are decreasing:  $\max(\vec{x}') \leq \max(\vec{x}) \rightarrow A_{\vec{x}} \subseteq A_{\vec{x}'}$ . We call  $\sum_{\varrho=1}^{r-1} i_\varrho = \ell$  and  $\mathcal{R}_{\ell+i} = \mathcal{S}_i$  for  $i = 1, \dots, i_r$ . By the preceding lemma and the induction hypothesis there are  $X_{1,i} \in \mathcal{R}_i$  and  $Y_{1,i} \in \mathcal{S}_i$  such that  $X_1 = X_{1,1} \times \dots \times X_{i_{r-1},1} \cap \Delta_\ell \in \bigotimes_{\varrho=1}^{r-1} \bigotimes_{i \in K_\varrho} \mathcal{R}_i$  and  $Y_1 = Y_{1,1} \times \dots \times Y_{i_r,1} \cap \Delta_{i_r} \in \bigotimes_{i=1}^{i_r} \mathcal{S}_i$  such that

- (1)  $\vec{x} < \vec{x}' \in X_1 \rightarrow \vec{x}' \in A_{\vec{x}}$ ,
- (2)  $X_{i,1} = f_{i_\varrho, i}[X_{i_\varrho, 1}]$  for  $i \in K_\varrho$ ,  $\varrho = 1, \dots, r-1$ ,

- (3) the elements of  $X_1, X_2, \dots, X_{r-1}$  alternate in a strong sense: there is an enumeration  $\langle \vec{z}_i : i < \omega \rangle$  of  $X_1 \cup \dots \cup X_r$  such that for any  $n$ ,  $\varrho = 1, \dots, r-1$ ,  $z_{rn+\varrho} \in X_\varrho$  and  $\max(z_{rn+\varrho}) < \min(z_{r(n+1)+\varrho})$
- (4)  $\vec{y} < \vec{y}' \in Y_1 \rightarrow \vec{y}' \in B_{\vec{y}}$ ,
- (5)  $Y_{i,1} = f_{r,1,i}[Y_{1,1}]$  for  $i \in K_r$ .

We take an interval partition  $\Pi = \langle \pi_n : n \in \omega \rangle$  such that in each  $\Pi$ -interval there is  $x_1 < \dots < x_k < y_1 < \dots < y_k$ ,  $x_i \in X_{i,1}$ ,  $y_i \in Y_{i,1}$ . We double the intervals in two different groupings: First take every two intervals and get  $\Psi$ . Let  $f_\Psi$  map the  $n$ -th  $\Psi$ -interval to  $n$ . Now take first just one  $\Pi$ -interval and then always two  $\Pi$ -intervals and get  $\Xi$ . Let  $f_\Xi$  map the  $n$ -th  $\Xi$ -interval to  $n$ . Then we apply non-near coherence and  $Q$ -pointness get  $X_{i,2} \subseteq X_{i,1}$ ,  $X_{i,2} \in \mathcal{R}_i$ , for  $i \in \bigcup_{\varrho=1}^{r-1} K_\varrho$ , and  $Y_{j,2} \subseteq Y_{j,1}$ ,  $Y_{j,2} \in \mathcal{S}_j$  for  $j \in K_r$ , pairwise disjoint hitting each  $\Psi$ -interval exactly once such that  $f_\Psi[X_{2,i}] \cap f_\Psi[Y_{2,i}] = \emptyset$  and  $f_\Xi[X_{2,i}] \cap f_\Xi[Y_{2,i}] = \emptyset$ . Now we fill up by enumerating alternately and repeat the five steps as in Blass' proof.  $\square$

Now we read the previous lemma just for one kind of grouping in the conclusion:  $\varrho = 2$  and  $i_1 = i_2 = k$ . However, this reading was not suitable for induction hence we wrote the previous lemma.

**Lemma 2.26.** (*Weak generalisation of [6, 1.2]*) *If  $\mathcal{R}_i, \mathcal{S}_i$ ,  $i = 1, \dots, k$ , are pairwise different selective ultrafilters and for any  $(i, j) \in k^2$ ,  $\mathcal{R}_i$  is not nearly coherent to  $\mathcal{S}_j$  and for  $\vec{x} \in [\omega]^k$  we have  $B_{\vec{x}} \in \bigotimes_{i=1}^k \mathcal{S}_i$  and for  $\vec{y} \in \omega^k$  we have  $A_{\vec{y}} \in \bigotimes_{i=1}^k \mathcal{R}_i$  then there are  $X \in \bigotimes_{i=1}^k \mathcal{R}_i$  and  $Y \in \bigotimes_{i=1}^k \mathcal{S}_i$  such that*

- (1)  $\forall \vec{x} \in X, \vec{y} \in Y (\max(\vec{x}) < \min(\vec{y}) \rightarrow \vec{y} \in A_{\vec{x}})$ ,
- (2)  $\forall \vec{x} \in X, \vec{y} \in Y (\max(\vec{y}) < \min(\vec{x}) \rightarrow \vec{x} \in B_{\vec{y}})$ ,
- (3) *the elements of  $X$  and  $Y$  alternate in a strong sense: there is an enumeration  $\langle \vec{z}_i : i < \omega \rangle$  of  $X \cup Y$  such that  $z_{2n} \in X$ ,  $z_{2n+1} \in Y$  and  $\max(\vec{z}_i) < \min(\vec{z}_{i+1})$ .*  $\square$

**Lemma 2.27.** *If  $\mathcal{R}_i, \mathcal{S}_i$ ,  $i = 1, \dots, k$ , are selective and each  $\mathcal{R}_i$  is not nearly coherent to any  $\mathcal{S}_j$  then*

$$\{A \times \omega^k : A \in \bigotimes_{i=1}^k \mathcal{R}_i\} \cup \{\omega^k \times B : B \in \bigotimes_{i=1}^k \mathcal{S}_i\} \cup \{\text{uppertriangle}^{2k}\}$$

generates  $\bigotimes_{i=1}^k \mathcal{R}_i \otimes \bigotimes_{i=1}^k \mathcal{S}_i$ .

*Proof.* Let  $Z \in \bigotimes_{i=1}^k \mathcal{R}_i \otimes \bigotimes_{i=1}^k \mathcal{S}_i$ . Now we apply the previous lemma to  $A_{\vec{y}} = \{\vec{x} : (\vec{x}, \vec{y}) \in Z\}$  and  $B_{\vec{x}} = \{\vec{y} : (\vec{x}, \vec{y}) \in Z\}$ .  $\square$

**Corollary 2.28.** *If  $\mathcal{R}_i, \mathcal{S}_i, i = 1, \dots, k$ , are selective ultrafilters and each  $\mathcal{R}_i$  is not nearly coherent to any  $\mathcal{S}_j$  then  $\bigotimes_{i=1}^k \mathcal{R}_i \otimes \bigotimes_{i=1}^k \mathcal{S}_i$  is the only ultrafilter  $\mathcal{V}$  over  $\omega^{2k}$  with projections  $\bigotimes_{i=1}^k \mathcal{R}_i$  and  $\bigotimes_{i=1}^k \mathcal{S}_i$  and uppertriangle $^{2k} \in \mathcal{V}$ .*

**Theorem 2.29.** *(Compare to [6, Theorem 2.1]). Let  $\mathcal{V}_1, \mathcal{V}_2$  be any two ultrafilters over  $\omega^k$ . Then there exists an ultrafilter  $\mathcal{U}$  over  $\mathbb{F}^k$  such that  $\mathcal{U} \dot{\cup} \mathcal{U} = \mathcal{U}$  and  $\hat{\min}_k(\mathcal{U}) = \mathcal{V}_1$ ,  $\hat{\max}_k(\mathcal{U}) = \mathcal{V}_2$ .*

*Proof.* As in Blass' [6, Theorem 2.1], using the Ellis-Namakura-Lemma [16, 32].  $\square$

**Lemma 2.30.** *Let  $\mathcal{R}_i, \mathcal{S}_i, i = 1, \dots, k$ , be selective ultrafilters and each  $\mathcal{R}_i$  be not nearly coherent to any  $\mathcal{S}_j$  and set  $\mathcal{V}_1 = \bigotimes_i \mathcal{R}_i$  and  $\mathcal{V}_2 = \bigotimes_j \mathcal{S}_j$ . Let  $\bar{a} \in \mathbb{F}^k$  be such that  $\min_k[\bar{a}] = \prod_k X_i \cap \Delta_k \in \mathcal{V}_1$  and  $\max_k[\bar{a}] = \prod_k Y_i \cap \Delta_k \in \mathcal{V}_2$  with  $X_i \in \mathcal{R}_i, Y_j \in \mathcal{S}_j$ . Let  $\mathcal{E}, \mathcal{W}$  be ultrafilters over  $\omega$  that are not nearly coherent to any of the  $\mathcal{R}_i, \mathcal{S}_j$ . Let  $f$  be finite-to-one. Then there are  $E \in \mathcal{E}, W \in \mathcal{W}$  and  $\bar{b} \sqsubseteq \bar{a}$  such that  $\min_k[\bar{b}] \in \mathcal{V}_1$  and  $\max_k[\bar{b}] \in \mathcal{V}_2$  and  $f[\bigcup\{\text{dom}(b_n) : n \in \omega\}] \cap f[E] = \emptyset$  and  $f[\bigcup\{\text{dom}(b_n) : n \in \omega\}] \cap f[W] = \emptyset$ .*

*Proof.* Let  $f$  be increasing and surjective.  $f^{-1}[\{i\}]$  is called the  $i$ -th  $f$ -interval. After combining  $f$ -intervals and thinning out  $\bar{a}$  (using that any of the  $\mathcal{R}_i, \mathcal{S}_j$  are ultrafilters) we may assume that the  $f^{-1}[\{i\}]$  are so wide that each  $\text{dom}(a_n)$  is a subset of some  $f^{-1}[\{i\}]$ . There are  $X_i^1 \subseteq X_i, X_i^1 \in \mathcal{R}_i, E \in \mathcal{E}, W \in \mathcal{W}$  with  $f[X_i^1] \cap f[E], f[W] = \emptyset$ . There are  $Y_i^1 \subseteq Y_i, Y_i^1 \in \mathcal{S}_i, E' \in \mathcal{E}, W' \in \mathcal{W}, E' \subseteq E, W' \subseteq W$ , with  $f[X_i^1] \cap f[E'], f[W'] = \emptyset$ . We choose  $\bar{b}^1 \in \mathcal{U}, \bar{b}^2 \in \mathcal{U}$  with  $\bar{b}^2 \sqsubseteq \bar{b}^1$ , such that

$$\begin{aligned} \bar{b}^1 &= \{a_n : \min_k(a_n) \in \prod X_i \cap \Delta_k\}, \\ \bar{b}^2 &= \{b_n^1 : \max_k(b_n^1) \in \prod Y_i \cap \Delta_k\}. \end{aligned}$$

Then  $\bar{b}^2$  is as desired.  $\square$

Now we use Corollary 2.28 in a crucial way:

**Lemma 2.31.** *(See [6, Theorem 2.2]) Given selective ultrafilters  $\mathcal{R}_i, \mathcal{S}_j$  such that any  $\mathcal{R}_i$  is not nearly coherent to any of the  $\mathcal{S}_j$ , we let  $\mathcal{V}_1 = \mathcal{R}_1 \otimes \dots \otimes \mathcal{R}_k$  and  $\mathcal{V}_2 = \mathcal{S}_1 \otimes \dots \otimes \mathcal{S}_k$  and let  $\mathbb{F}^k$  be partitioned into finitely many pieces. Then we can find  $\bar{a} \in (\mathbb{F}^k)^\omega$  with  $\min_k[\bar{a}] \in \mathcal{V}_1$  and  $\max_k[\bar{a}] \in \mathcal{V}_2$  such that  $\text{FU}(\bar{a})$  is included in one part of the partition.*

*Proof.* Let  $\mathcal{U}$  be an idempotent ultrafilter with  $\hat{\min}(\mathcal{U}) = \mathcal{V}_1$  and  $\hat{\max}(\mathcal{U}) = \mathcal{V}_2$  as in Theorem 2.29. Now we go on literally as in Blass' proof: We take a piece  $P$  of the partition with  $P \in \mathcal{U}$ . We call a set  $s \in \mathbb{F}^k$  good if  $s \in P$  and for  $\mathcal{U}$ -most  $t, s \cup t \in P$ . By idempotence of  $\mathcal{U}$ , this implies that for  $\mathcal{U}$ -most  $t$ , for  $\mathcal{U}$ -most  $u, s \cup t \cup u \in P$ , and therefore for  $\mathcal{U}$ -most  $t, s \cup t$  is good. Since

$P \in \mathcal{U}$  the idempotence of  $\mathcal{U}$  also implies that  $\mathcal{U}$ -most  $s$  are good. We let

$$Z_n = \{t \in \mathbb{F}^k : t \text{ is good and for all good } s \\ \text{with } \max(\text{dom}(s)) \leq n, s \cup t \text{ is good}\}.$$

$Z_n$  is in  $\mathcal{U}$  because for each  $n$  there are only finitely many  $s$  with  $\max(\text{dom}(s)) \leq n$  and  $\mathcal{U}$  is closed under finite intersection. For each  $n \in \omega$  let

$$A_n = \{\vec{x} \in \omega^k : \min(\vec{x}) > n \text{ and for } \mathcal{V}_2\text{-most } \vec{y} \\ \text{there exists } t \in Z_n \text{ with } \min_k(t) = \vec{x} \text{ and } \max_k(t) = \vec{y}\}.$$

We show that  $A_n \in \mathcal{V}_1$ . We ignore the clause  $\min(\vec{x}) > n$  since  $\mathcal{V}_1$  is a product of non-principal ultrafilters. We show that for  $\mathcal{V}_1$ -most  $\vec{x}$  for  $\mathcal{V}_2$ -most  $\vec{y}$  there exists  $t \in Z_n$  with  $\min_k(t) = \vec{x}$ ,  $\max_k(t) = \vec{y}$ . We write  $(\min_k, \max_k)$  for the function  $\mathbb{F}^k \rightarrow \omega^{2k}$  sending each  $t$  to  $(\min_k(t), \max_k(t))$ . The statement to be proved is  $(\min_k, \max_k)[Z_n] \in \mathcal{V}_1 \otimes \mathcal{V}_2$ . Since  $Z_n \in \mathcal{U}$  it suffices to show  $(\hat{\min}_k, \hat{\max}_k)(\mathcal{U}) = \mathcal{V}_1 \otimes \mathcal{V}_2$ . Remember,  $(\hat{\min}_k, \hat{\max}_k)(\mathcal{U})$  is defined in Def. 2.9 (4).  $(\hat{\min}_k, \hat{\max}_k)(\mathcal{U})$  is an ultrafilter of  $\omega^{2k}$  with the projections onto the first  $k$  coordinates  $\mathbf{V}_1 = \bigotimes_{i=1}^k \mathcal{R}_i$  and the projection onto the last  $k$  coordinates  $\mathcal{V}_1 = \bigotimes_{i=1}^k \mathcal{S}_i$ .  $(\hat{\min}_k, \hat{\max}_k)(\mathcal{U})$  contains uppertriangle<sup>2k</sup>. So by Corollary 2.28 we have  $(\hat{\min}_k, \hat{\max}_k)(\mathcal{U}) = \mathcal{V}_1 \otimes \mathcal{V}_2$ . Thus  $(\hat{\min}_k, \hat{\max}_k)(\mathcal{U}) = \mathcal{V}_1 \otimes \mathcal{V}_2$ .

Hence we have  $A_n \in \mathcal{V}_1$ .

The  $Z_n$  are decreasing and therefore also the sequence  $A_n$ ,  $\bigcap A_n = \emptyset$ , so each  $\vec{x}$  belongs to a largest  $A_n$ . For this  $n$  let

$$B_{\vec{x}} = \{\max_k(t) : t \in Z_n \text{ and } \min_k(t) = \vec{x}\}$$

and  $B_{\vec{x}} \in \mathcal{V}_2$  by the definition of  $A_n$ . We let  $A_{\vec{y}} = A_{\max(\vec{y})}$ . Now be Lemma 2.26 we find  $X \in \mathcal{V}_1$  and  $Y \in \mathcal{V}_2$  such that

- (1)  $\forall \vec{x} \in X, \vec{y} \in Y (\max(\vec{y}) < \min(\vec{x}) \rightarrow \vec{x} \in A_{\vec{y}}),$
- (2)  $\forall \vec{x} \in X, \vec{y} \in Y (\max(\vec{x}) < \min(\vec{y}) \rightarrow \vec{y} \in B_{\vec{x}}),$
- (3) the elements of  $X$  and  $Y$  alternate in a strong sense: there is an enumeration  $\langle z_i : i < \omega \rangle$  of  $X \cap Y$  such that  $z_{2n} \in X$ ,  $z_{2n+1} \in Y$  and  $\max(\vec{z}_i) < \min(\vec{z}_{i+1})$ .

By induction on  $p \in \omega$  we choose  $s_p \in \mathbb{F}^k$  in such a way that every union of finitely many  $s_p$ 's is good,  $\min_k(s_p) = x_p \in X$  and  $\max_k(s_p) = y_p \in Y$ . Suppose that  $s_0, \dots, s_{p-1}$  have already been chosen. Since  $x_p \in A_{y_{p-1}}$  (by (1)) the  $n$  involved in the definition of  $B_{x_p}$  is a least  $\max(y_{p-1})$ , that is  $B_{x_p} = \{\max_k(t) : t \in Z_n \text{ and } \min_k(t) = x_p\}$  for some  $n \geq \max(y_{p-1})$ . Since the  $Z_n$  are decreasing and since  $y_p \in B_{x_p}$  by (2) we infer that there is  $t \in Z_{y_{p-1}}$  with  $\min_k(t) = x_p$  and  $\max_k(t) = y_p$ . The definition of  $Z_{y_{p-1}}$  says that  $t$  is good and that for every union  $s$  of some of the functions  $s_0, \dots, s_{p-1}$ , since  $s$  is good and  $\max(\max_k(s)) \leq \max(y_{p-1})$ ,  $s \cup t$  is good. Thus  $t$  can serve as  $s_p$ . The sequence  $S = \{s_0, s_1, \dots\} \in (\mathbb{F}^k)^\omega$  just constructed

has  $\min_k[S] = X \in \mathcal{V}_1$  and  $\max_k[S] = Y \in \mathcal{V}_2$  and  $\text{FU}(S) \subseteq P$ .  $\square$

We write a self-strengthening that is combined with lemma 2.30:

**Lemma 2.32.** (See [6, Theorem 2.2]) *Given selective ultrafilters  $\mathcal{R}_i, \mathcal{S}_j$  such that any  $\mathcal{R}_i$  is not nearly coherent to any of the  $\mathcal{S}_j$ , we let  $\mathcal{V}_1 = \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_k$  and  $\mathcal{V}_2 = \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_k$  and let  $\bar{a} \in (\mathbb{F}^k)^\omega$  with  $\min[\bar{a}] \in \mathcal{V}_1$  and  $\max[\bar{a}] \in \mathcal{V}_2$  be partitioned into finitely many pieces. In addition let  $\mathcal{W}$  be an ultrafilter that is not nearly coherent with any of the  $\mathcal{R}_i, \mathcal{S}_j$ , and let  $f$  be a finite-to-one function. Then we can find  $W \in \mathcal{W}$  and  $\bar{b} \in (\mathbb{F}^k)^\omega$  with  $\bar{b} \sqsubseteq \bar{a}$  with  $\min_k[\bar{b}] \in \mathcal{V}_1$  and  $\max_k[\bar{b}] \in \mathcal{V}_2$  such that  $\text{FU}(\bar{b})$  is included in one part of the partition and  $f[W] \cap f[\bigcup\{\text{dom}(b_n) : n \in \omega\}] = \emptyset$ .*

*Proof.* First we look for  $\bar{b}' \sqsubseteq \bar{a}$  such that  $\text{FU}(\bar{b}')$  is included in one part of the partition. The existence of such a  $\bar{b}'$  is proved as in Blass' work. Then we take  $\bar{b}' \sqsubseteq \bar{b}$  as in Lemma 2.30.  $\square$

Now finally Theorem 2.12 is proved by an induction of length  $\omega_1$ : We enumerate the partitions of  $\mathbb{F}^k$  into two sets with the aid of CH and we enumerate all finite-to-one functions with the aid of CH. We construct by induction on  $\varepsilon < \omega_1$  a  $\sqsubseteq^*$ -descending sequence  $\langle \bar{c}^\varepsilon : \varepsilon < \omega_1 \rangle$  in  $(\mathbb{F}^k)^\omega$  that generates an Milliken–Taylor ultrafilter. We take a diagonal  $\sqsubseteq^*$ -lower bound in the limit steps as in Blass' proof of [6, Theorem 2.4] such that  $\min_k[\bar{c}^\varepsilon] \in \mathcal{V}_1$  and  $\max_k[\bar{c}^\varepsilon] \in \mathcal{V}_2$ , (this is not trivial and hinges on the countable cofinality, see Prop. 6.2) and we use Lemma 2.31 with the partition and the finite-to-one function given by the enumeration in the successor steps.  $\square_{2.12}$

At the very end of the paper we prove (Prop. 6.2) that the continuum hypothesis is really needed in Theorem 2.13. It is the limit step of cofinality  $\omega_1$  in the analogous version of the above proof that fails in general: In general, only descending sequences whose length has countable cofinality can have a diagonal lower bound.

### 3. FORCING

In the section we explain the use of Milliken–Taylor ultrafilters in our forcing.

**Definition 3.1.** *In the  $k$ -coloured Matet forcing,  $\mathbb{M}$ , the conditions are pairs  $(a, \bar{c})$  such that  $a \in \mathbb{F}^k$  and  $\bar{c} \in ((\mathbb{F})^k)^\omega$  and  $a < c_0$ . The forcing order is  $(b, \bar{d}) \geq (a, \bar{c})$  (recall the stronger condition is the larger one) iff  $a \subseteq b$  and  $b \setminus a$  is a concatenation of finitely many of the  $c_n$  and  $\bar{d}$  is a condensation of  $\bar{c}$ .*

**Definition 3.2.** *Given a centred system  $\mathcal{C} \subseteq ((\mathbb{F})^k)^\omega$ , the notion of forcing  $\mathbb{M}(\mathcal{C})$  consists of all pairs  $(s, \bar{a})$  such that  $\bar{a} \in \mathcal{C}$ . The forcing order is the same as in the Matet forcing. In the special case that  $\mathcal{C}$  is the set of members*

of a  $\sqsubseteq^*$ -descending sequence  $\langle \bar{c}^\eta : \eta < \beta \rangle$ , and their  $=^*$  equivalent elements, we also write  $\mathbb{M}(\bar{c}^\eta : \eta < \beta)$  for  $\mathbb{M}(\mathcal{C})$ .

In this section we use  $\mathbb{M}(\bar{a}^\eta : \eta < \beta)$  for  $\sqsubseteq^*$ -descending sequences of length 1, of length  $< \kappa$  and of length  $\kappa$  where  $\kappa = (2^\omega)^\mathbf{V}$  is assumed to be regular. We write  $\text{set}(\bar{a})(i)$  for  $\bigcup \{a_n^{-1}[\{i\}] : n < \omega\}$  and  $\text{set}(\bar{a}) = \bigcup_{i=1}^k \text{set}(\bar{a})(i)$ . For every  $i = 1, \dots, k$ , the forcing  $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$  diagonalises (“shoots a real through”)  $\{\text{set}(\bar{a}^\alpha)(i) : \alpha < \beta\}$ , namely  $\mu(i)$  the  $i$ -fibre of the generic real is a pseudointersection of this set. Here we use

**Definition 3.3.** Let  $G$  be  $\mathbb{M}(\bar{a}^\eta : \eta < \beta)$ -generic over  $\mathbf{V}$ .

$$\mu(i) = \bigcup \{w^{-1}[\{i\}] : \exists \bar{a}(w, a) \in G\},$$

is called the  $i$ -fibre of the  $\mathbb{M}(\bar{a}^\eta : \eta < \beta)$ -generic real.

Given a Milliken–Taylor ultrafilter  $\mathcal{U}$  on  $\mathbb{R}^k$  we let  $\mathbb{M}(\mathcal{U})$  be  $\mathbb{M}(\mathcal{C})$  for the special centered system  $\mathcal{C} = \mathcal{U}$ .

The following property of stable ordered-union ultrafilters  $\mathcal{U}$  will be important for our proof:

**Theorem 3.4.** (Eisworth [15, “ $\rightarrow$ ” Theorem 4, “ $\leftarrow$ ” Cor. 2.5, this direction works also with non- $P$  ultrafilters]) Let  $\mathcal{U}$  be a stable ordered-union ultrafilter over  $\mathbb{R}$  and let  $\mathcal{W}$  be a  $P$ -point.  $\mathcal{W} \not\leq_{RB} \Phi(\mathcal{U})$  iff  $\mathcal{W}$  continues to generate an ultrafilter after we force with  $\mathbb{M}(\mathcal{U})$ .

With the same prove one shows:

**Theorem 3.5.** Let  $\mathcal{U}$  be a stable ordered-union ultrafilter over  $\mathbb{R}^k$  and let  $\mathcal{W}$  be a  $P$ -point.  $\mathcal{W} \not\leq_{RB} \Phi(\mathcal{U})$  iff  $\mathcal{W}$  continues to generate an ultrafilter after we force with  $\mathbb{M}(\mathcal{U})$ .

If a  $\sqsubseteq^*$ -descending sequence  $\langle \bar{c}_\varepsilon : \varepsilon < \omega_1 \rangle$  has the property that  $\{\text{FU}(\bar{c}_\varepsilon) : \varepsilon < \omega_1\}$ , generates an ultrafilter  $\mathcal{U}$  over  $\mathbb{R}^k$ , then  $\mathcal{U}$  is a stable ordered-union ultrafilter and  $\mathbb{M}(\mathcal{U}) = \mathbb{M}(\bar{c}_\varepsilon : \varepsilon < \omega_1)$ . Only this type of Milliken–Taylor ultrafilters appears in this paper along the construction. By Theorem 2.12 we can chose the  $\bar{c}_\varepsilon$  such that  $\Phi(\mathcal{U}) \not\leq_{RB} \mathcal{E}$ . Then this is a stable ordered-union ultrafilter and  $\mathbb{M}(\mathcal{U}) = \mathbb{M}(\bar{c}_\varepsilon : \varepsilon < \omega_1)$ . So much about preserving an ultrafilter.

We remark, though it is never used in the proof of the main theorem:

**Theorem 3.6.** Suppose that  $\mathcal{U}$  is an Milliken–Taylor ultrafilter and  $\mathcal{F}$  is a filter over  $\omega$  and  $f$  is finite-to-one and  $\Phi(\mathcal{U}) \leq_{RB} \mathcal{F}$ . Then  $\mathbb{M}(\mathcal{U})$  forces that  $\mathcal{F}$  is not ultra by finite-to-one.

*Proof.* Let  $f \in \mathbf{V}$  be finite-to-one such that  $f(\Phi(\mathcal{U})) \subseteq f(\mathcal{F})$ . Let  $G$  be  $\mathbb{M}(\mathcal{U})$ -generic over  $\mathbf{V}$ . Let  $g$  be a name for a finite to one function in  $\mathbf{V}^{\mathbb{M}(\mathcal{U})}$ . We let

$$\mu = \bigcup \{t : \exists \bar{a}(t, \bar{a}) \in G\}$$



and let  $\mu$  be a name for  $\mu$ . We show:

$$\begin{aligned} \forall (s, \bar{a}) \in \mathbb{M}(\mathcal{U}) * \mathbb{Q} \forall Y \in \mathcal{F} \exists (t, \bar{b}) \geq_{\mathbb{M}(\mathcal{U})} (s, \bar{a}) \\ (t, \bar{b}) \Vdash_{\mathbb{M}(\mathcal{U})} g[f[Y]] \cap g[f[\mu]] \neq \emptyset \wedge g[f[Y]] \cap (\omega \setminus g[f[\mu]]) \neq \emptyset \end{aligned}$$

This suffices. Let  $(s, \bar{a})$  and  $Y$  be given. Since  $f[\text{set}(\bar{a})] \in f(\Phi(\mathcal{U})) \subseteq f(\mathcal{F})$ , we have  $f[\text{set}(\bar{a})] \cap f[Y]$  is infinite. So there is  $t \in \text{FU}(\bar{a})$  such that  $f[t] \cap f[Y] \neq \emptyset$ . So  $(s \cup t, \bar{a}; \text{past } t) \Vdash_{\mathbb{M}(\mathcal{U})} g[f[Y]] \cap g[f[\mu]] \neq \emptyset$ . For  $r < u < v$  we call  $(r, u, v)$  good for avoiding  $Y$  if

$$\begin{aligned} (r \cup u \cup v, \bar{a}; \text{past } v) \Vdash_{\mathbb{M}(\mathcal{U})} g[f[r]] < g[f[u]] \\ \wedge (\max(g[f[r]]), \min(g[f[u]])) \cap g[f[Y]] \neq \emptyset. \end{aligned}$$

Next we define a colouring  $h$  of  $[\text{FU}(\bar{a}; \text{past } t)]_{<}^3$  by

$$h(r < u < v) = \begin{cases} 1 & \text{if } (r, u, v) \text{ is good for avoiding } Y, \\ 0 & \text{else.} \end{cases}$$

Since  $\mathcal{U}$  is a Milliken–Taylor ultrafilter there is a monochromatic  $\bar{b} \sqsubseteq (\bar{a}; \text{past } t)$ ,  $\bar{b} \in \mathcal{U}$ . Since  $g$  and  $f$  are finite-to-one, the colour is 1. so we have  $(s \cup t, \bar{b}) \Vdash_{\mathbb{M}(\mathcal{U})} g[f[Y]] \cap (\omega \setminus g[f[\mu]]) \neq \emptyset$ .  $\square$

We remark further, that taking  $\mathcal{F}^+$  instead of the destroyed  $\mathcal{F}$  gives a similar picture.

**Definition 3.7.** Let  $\mathcal{F}$  be a filter.  $\mathcal{F}^+ = \{X \in [\omega]^\omega : \forall Y \in \mathcal{F} Y \cap X \neq \emptyset\}$ .

**Lemma 3.8.** Let  $f$  be finite-to-one, increasing and surjective. Let  $\mathcal{F}$  be a filter. Then  $f(\mathcal{F}^+) = (f(\mathcal{F}))^+$ .

*Proof.* “ $\subseteq$ ”: Let  $X \in f(\mathcal{F}^+) = \{f[Y] : Y \in \mathcal{F}^+\}$ . Then there is  $Y \in \mathcal{F}^+$ ,  $X = f[Y]$ . Let  $Z \in f(\mathcal{F})$  be any element, say  $Z = f[U]$ ,  $U \in \mathcal{F}$ . We have  $X \cap Z = f[Y] \cap f[U] \supseteq f[Y \cap U]$  is infinite, since  $f$  is finite-to-one and  $Y \cap U \in [\omega]^\omega$ .

“ $\supseteq$ ”: Let  $X \in (f(\mathcal{F}))^+$ . Then for any  $Z \in f(\mathcal{F})$ ,  $X \cap Z \in [\omega]^\omega$ . Let  $U = f^{-1}[X]$ . We have  $X = f[U]$ . Now  $U \in \mathcal{F}^+$ , since for any  $Y \in \mathcal{F}$ ,  $Y = f^{-1}[f[Y]]$  and  $U \cap f[Y] = f^{-1}[X] \cap f^{-1}[f[Y]] = f^{-1}[X \cap f[Y]]$  is infinite, since  $X \cap f[Y]$  is infinite and  $f$  is surjective.  $\square$

**Observation 3.9.** Let  $f$  be finite-to-one, increasing and surjective. Let  $\mathcal{F}$  be a filter. Then  $f(\mathcal{F}^+)$  is ultra iff  $f(\mathcal{F})$  is ultra.

*Proof.*  $\mathcal{W}^+ = \mathcal{W}$  iff  $\mathcal{W}$  is ultra.  $f(\mathcal{F}^+) = (f(\mathcal{F}))^+ = f(\mathcal{F})$  iff  $f(\mathcal{F})$  is ultra and  $f(\mathcal{F}^+) = (f(\mathcal{F}))^+ = (f(\mathcal{F}))^{++} = f(\mathcal{F})$  iff  $(f(\mathcal{F}))^+$  is ultra.

#### 4. GENERATING NEW SELECTIVE ULTRAFILTERS EXTENDING THE DESTROYED ONES

Now there are two tasks:

The successor task: Given  $\mathcal{R}_{i,\alpha} \in \mathbf{V}_\alpha$ ,  $i = 1, \dots, k$ , and possibly an ultrafilter  $\mathcal{W}$  handed down by a diamond, and the  $P$ -point  $\mathcal{E}$  from the ground model, we choose  $\mathcal{S}_{i,\alpha}$ ,  $i = 1, \dots, k$ , and  $\mathcal{U}_\alpha$  in  $\mathbf{V}_\alpha$  according to Theorem 2.12. Then we force with  $\mathbb{M}(\mathcal{U}_\alpha)$  and thus get  $\mathbf{V}_{\alpha+1}$ . We show how we get a selective ultrafilters  $\mathcal{R}_{i,\alpha+1} \supseteq \mathcal{R}_{i,\alpha} \cup \{\mu_\alpha(i)\}$ .  $i = 1, \dots, k$  in  $\mathbf{V}_{\alpha+1} = \mathbf{V}_\alpha^{\mathbb{M}(\mathcal{U}_\alpha)}$  that are not nearly coherent to  $\mathcal{E}$  nor to  $\mathcal{W}$  and such that for  $i \neq j$ ,  $\mathcal{R}_{i,\alpha}$  is not nearly coherent to  $\mathcal{R}_{j,\alpha}$ . Indeed, there is only one choice for the ultrafilters if we impose that for  $i = 1, \dots, k$ , the  $i$ -fibre of the Matet generic real for  $\mathbb{M}(\mathcal{U}_{i,\alpha})$  is in  $\mathcal{R}_{i,\alpha+1}$ .

The limit tasks: Suppose that  $\alpha \leq \omega_2$  is a limit ordinal. If  $\text{cf}(\alpha) > \omega$ , we can just take  $\mathcal{R}_{i,\alpha} = \bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma}$  and the inductive hypotheses (which will be stated precisely below) all but CH will be carried on. So we concentrate on the hard case,  $\text{cf}(\alpha) = \omega$ . Since  $\mu_\gamma(i) \in \mathcal{R}_{i,\gamma+1}$ , we have  $\bigcup \mathcal{R}_{i,\gamma} = \bigcup_{\gamma < \alpha} (\mathcal{R}_{i,\gamma} \cup \{\mu_\gamma(i)\})$ . Given  $\mathcal{R}_{i,\beta}$ ,  $\beta < \alpha$  for  $i = 1, \dots, k$ , in  $\mathbf{V}_\alpha$  chosen according to the induction hypotheses, we find that  $(\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma})^{+\alpha}$  is a selective coideal and that there are suitable selective ultrafilters  $\mathcal{R}_{i,\alpha} \in \mathbf{V}^{\mathbb{P}_\alpha}$  extending  $\mathcal{R}_{i,\gamma}$ ,  $\gamma < \alpha$  for  $i = 1, \dots, k$ , all not nearly coherent to  $\mathcal{E}$  and for  $i \neq j$  not to  $\mathcal{R}_{j,\alpha}$ , and such that the  $\mathcal{R}_{i,\alpha}$  preserve the induction hypotheses. This time the choice of the  $\mathcal{R}_{i,\alpha}$  is not unique.

It turned out that there is a relation  $\bar{R}$  such that  $\mathbb{P}_\alpha$  is  $(\mathcal{S}, \bar{R}, \bar{g})$ -preserving is the notion we want to carry in the inductive choice of the iteration. We do not know whether one can carry on selective coideals alone.

For the limit step  $\alpha$  of countable cofinality, the following lemma explains why  $(\bigcup \mathcal{R}_{i,\gamma})^{+\alpha}$  is a superset of any candidate for  $\mathcal{R}_{i,\alpha}$ :

**Lemma 4.1.** *Let  $\mathcal{F}$  be a filter over  $\omega$  and let  $\mathcal{R}$  be an ultrafilter over  $\omega$ .  $\mathcal{R} \subseteq \mathcal{F}^+$  iff  $\mathcal{R} \supseteq \mathcal{F}$ .*

*Proof.* Suppose that  $X \in \mathcal{F} \setminus \mathcal{R}$ . Then, as  $\mathcal{R}$  is ultra,  $\omega \setminus X \in \mathcal{R}$ , and hence  $\mathcal{R} \not\subseteq \mathcal{F}^+$ .

For the other direction, if there is  $X \in \mathcal{R}$  and  $Y \in \mathcal{F}$  with  $X \cap Y = \emptyset$ , then  $\omega \setminus Y \in \mathcal{R}$  and hence  $\mathcal{R} \not\supseteq \mathcal{F}$ .  $\square$

*Remark 4.2.* Throughout Sections 4, 5, 6, in lieu of  $\Phi(\mathcal{U}_\gamma(i)) \subseteq \mathcal{R}_{i,\gamma}$  we could work with the weaker  $\Phi(\mathcal{U}_\gamma(i)) \leq_{\text{RB}} \mathcal{R}_{i,\gamma}$  and see that our results here are really complementary to Eisworth's result 3.5. However, since this necessitates variables for the finite to-one-functions witnessing the Rudin–Blass order relation  $\leq_{\text{RB}}$  we stick to the easier notation.

We name the generic reals:

**Definition 4.3.** (1) *Let  $\mathbb{M}(\mathcal{U})$  be the Matet order with an Milliken–Taylor ultrafilter  $\mathcal{U}$  over  $\mathbb{F}^k$  and let  $G$  be  $\mathbb{M}(\mathcal{U})$ -generic over  $\mathbf{V}$ . We let*

$$\mu = \bigcup \{\text{dom}(w) : \exists \bar{a} \in \mathcal{U}(w, \bar{a}) \in G\}$$

and for  $i = 1, \dots, k$ ,

$$\mu(i) = \bigcup \{w^{-1}[\{i\}] : \exists \bar{a} \in \mathcal{U}_\beta(w, \bar{a}) \in G\}.$$

$\mu(i)$  is the Matet generic real for colour  $i$ .

- (2) Let  $\mathbb{P} = \langle \mathbb{P}_\gamma, \mathbb{M}(\mathcal{U}_\beta) : \gamma \leq \alpha, \beta < \alpha \rangle$  be an iteration of Matet forcings with Milliken–Taylor ultrafilters  $\mathcal{U}_\alpha$  over  $\mathbb{F}^k$ . Let  $G^\beta$  be  $\mathbb{M}(\mathcal{U}_\beta)$ -generic over  $\mathbf{V}_\beta$ . We write

$$\mu_\beta = \bigcup \{\text{dom}(w) : \exists \bar{a} \in \mathcal{U}_\beta(w, \bar{a}) \in G^\beta\}$$

and for  $i = 1, \dots, k$ ,

$$\mu_\beta(i) = \bigcup \{w^{-1}[\{i\}] : \exists \bar{a} \in \mathcal{U}_\beta(w, \bar{a}) \in G^\beta\}.$$

We direct the extensions a bit by adding  $\mu_\gamma(i) \in \mathcal{R}_{i, \gamma+1}$  to the hypotheses we carry on.

In addition to preserving  $\bar{R}$  in the successor step there is the phenomenon that the selective ultrafilter  $\mathcal{R}_{i, \alpha+1}$  is already given by

$$(4.1) \quad \mathcal{R}_{i, \alpha+1} = \text{fil}(\mathcal{R}_{i, \alpha} \cup \{\mu_\alpha(i)\})^{\alpha+1}$$

So we do not have to construct it. Equation (4.1) reminds Mathias' work on retrieving the ultrafilter  $\mathcal{R}$  in the ground model from the generic real in Mathias forcing with  $\mathcal{R}$ ; here we get a selective ultrafilter in the extension by combining the  $\mathbb{M}(\mathcal{U})$ -generic real with any selective ultrafilter  $\mathcal{R}$  with  $\Phi(\mathcal{U}) \subseteq \mathcal{R}$ . Note that  $\mu_\alpha(i)$  does not diagonalise  $\mathcal{R}_{i, \alpha}$ , it only diagonalises  $\Phi(\mathcal{U}_\alpha(i))$ .

**Definition 4.4.** The step function defined from the inverse of the increasing enumeration of the set  $\mu_\alpha$  is called  $f_{\mu_\alpha}$ , so  $f_{\mu_\alpha}(n) = |\mu_\alpha \cap n|$ . In general we define for  $E \in [\omega]^\omega$   $f_E$  by  $f_E(n) = |E \cap n|$ .

$f_E$  is finite-to-one weakly increasing and surjective.

**Theorem 4.5.** The successor step. Let  $\mathcal{E}$  be a  $P$ -point and let  $\mathcal{W}$  be an ultrafilter. Let for  $i = 1, \dots, k$   $\mathcal{R}_i$  be a selective ultrafilter, such that  $\mathcal{R}_i$  is not isomorphic to  $\mathcal{R}_j$ , and  $\mathcal{R}_i$  not nearly coherent to  $\mathcal{E}, \mathcal{W}$ . Let  $\mathcal{U}$  is a Milliken–Taylor ultrafilter over  $\mathbb{F}^k$ , such that  $\Phi(\mathcal{U}(i)) \subseteq \mathcal{R}_i$  and  $\Phi(\mathcal{U}) \not\leq_{\text{RB}} \mathcal{E}, \mathcal{W}$ . Then after forcing with  $\mathbb{M}(\mathcal{U})$  we have

- (1a)  $\text{fil}(\mathcal{R}_i \cup \{\mu(i)\})^{\mathbb{M}(\mathcal{U})}$  is a selective ultrafilter and
- (1b)  $\text{fil}(\mathcal{R}_i \cup \{\mu(i)\})^{\mathbb{M}(\mathcal{U})}$  and  $\text{fil}(\mathcal{R}_j \cup \{\mu(j)\})^{\mathbb{M}(\mathcal{U})}$  are not isomorphic and
- (1c)  $\text{fil}(\mathcal{R}_i \cup \{\mu(i)\})^{\mathbb{M}(\mathcal{U})}$  and  $\mathcal{E}$  (pedantically  $\text{fil}((\mathcal{E})^{\mathbb{M}(\mathcal{U})})$ ) are not nearly coherent and  $f_\mu(\mathcal{E}) = f_\mu(\mathcal{W})$ .

*Proof.* We first cite some thinning out lemmata.

**Definition 4.6.**  $(t, \bar{b})$  is called a pure extension of  $(s, \bar{a})$  iff  $s = t$ .

**Lemma 4.7.** ([15, Lemma 2.6])  $\mathbb{M}(\mathcal{U})$  has the pure decision property, that is, for any  $\varphi$  in the forcing language for any  $(s, \bar{a}) \in \mathbb{M}(\mathcal{U})$ , there is  $\bar{b} \in \mathcal{U}$ ,  $\bar{b} \sqsubseteq \bar{a}$  such that  $(s, \bar{b})$  decides  $\varphi$ .  $\square$

Also the following definition is given by Eisworth [15]:

**Definition 4.8.** (1) Let  $\underline{A}$  be a name for an infinite subset of  $\omega$  (that means, the weakest condition forces this).  $(s, \bar{b})$  is neat for  $\underline{A}$  iff:

$$(4.2) \quad \begin{aligned} & (\forall t \in \text{FU}(\bar{b}))(\forall i \in \omega)(\exists \text{truth}(i, t) \in \{\text{true}, \text{false}\}) \\ & (\forall r \in \text{FU}(\bar{b} ; \text{past } t))(\forall i \leq \max(r)) \\ & ((s \cup t \cup r, (\bar{b} ; \text{past } r)) \text{ decides } i \in \underline{A} \text{ with truth } \text{truth}(i, t)). \end{aligned}$$

(2) Let  $\langle \underline{A}_j : j < \omega \rangle$  be a sequence of names for infinite subsets of  $\omega$  that are descending in the  $\sqsubseteq$ -order.  $(s, \bar{b})$  is neat for  $\langle \underline{A}_j : j < \omega \rangle$  iff:

$$(4.3) \quad \begin{aligned} & (\forall t \in \text{FU}(\bar{b}))(\forall i, j \in \omega)(\exists \text{truth}(i, j, t) \in \{\text{true}, \text{false}\}) \\ & (\forall r \in \text{FU}(\bar{b} ; \text{past } t))(\forall i, j \leq \max(r)) \\ & ((s \cup t \cup r, (\bar{b} ; \text{past } r)) \text{ decides } i \in \underline{A}_j \text{ with truth } \text{truth}(i, j, t)) \end{aligned}$$

(3) If  $h$  is a finite-to-one surjective weakly increasing function then  $h(i) \leq i$  for  $i \in \omega$ . Hence for  $h \upharpoonright i$  there are only finitely many possibilities and we can use the notion of neatness again:  $(s, \bar{b})$  is neat for  $\underline{h}$  iff:

$$(4.4) \quad \begin{aligned} & (\forall t \in \text{FU}(\bar{b}))(\forall i \in \omega)(\exists \hat{h}: i \rightarrow i) \\ & (\forall r \in \text{FU}(\bar{b} ; \text{past } t))(\forall i \leq \max(r)) \\ & ((s \cup t \cup r, (\bar{b} ; \text{past } r)) \Vdash \underline{h} \upharpoonright i = \hat{h}) \end{aligned}$$

**Lemma 4.9.** ([15, Lemma 2.7, Lemma 2.8])  $(s, \bar{a}) \in \mathbb{M}(\mathcal{U})$ . Let  $(\underline{X}, \langle \underline{A}_j : j < \omega \rangle, \underline{h})$  given such that the weakest condition forces:  $\underline{X}$  is a infinite subset of  $\omega$ ,  $\langle \underline{A}_j : j < \omega \rangle$  is a  $\sqsubseteq$ -descending sequence of subsets of  $\omega$ ,  $\underline{h}$  is a surjective weakly increasing finite-to-one function. Then there is  $\bar{b} \sqsubseteq \bar{a}$  such that  $(s, \bar{b})$  is neat for  $\underline{X}$  and  $\langle \underline{A}_j : j < \omega \rangle$  and  $\underline{h}$ .  $\square$

Proof of the successor theorem:

We fix  $i \in \{1, \dots, k\}$ . We first prove that  $\text{fil}(\mathcal{R}_i \cup \{\mu(i)\})$  is an ultrafilter. At this point that  $\Phi(\mathcal{U}(i)) \subseteq \mathcal{R}_i$  is used: By a density argument, for every  $Z \in \mathcal{R}_i$ ,  $Z \cap \mu(i)$  is infinite. So we have that  $\text{fil}(\mathcal{R}_i \cup \{\mu(i)\})$  is a filter base. It will be an ultrafilter once we show that for every  $A \subseteq \omega$  there is  $Y \in \text{fil}(\mathcal{R}_i \cup \{\mu(i)\})$  such that  $Y \subseteq A$  or  $Y \subseteq \omega \setminus A$ .

Now let  $\underline{A}$  so that  $\Vdash \text{“}\underline{A} \in [\omega]^\omega\text{”}$ . Let  $(s, \bar{a})$  be neat for  $\underline{A}$ . For  $t \in \text{FU}(\bar{a} ; \text{past } s)$ , we let

$$X_t := \{i \in \omega : (\forall r \in \text{FU}(\bar{a} ; \text{past } t))((s \cup t \cup r, \bar{a} ; \text{past } r) \Vdash i \in \underline{A})\}.$$

By Taylor's theorem, applied to  $f: [\text{FU}(\bar{a} ; \text{past } s)]_{<}^2 \rightarrow 2$  with  $f(t_1, t_2) = \text{yes/no}$  if  $(s \cup t_1 \cup t_2, \bar{a} ; \text{past } t_2) \Vdash X_{t_1} \in / \notin \mathcal{R}_i$  and the Milliken–Taylor ultrafilter  $\mathcal{U}$ , there is  $\bar{a}' \sqsubseteq \bar{a} ; \text{past } s$ ,  $\bar{a}' \in \mathcal{U}$  such that for  $(t, r) \in [\text{FU}(\bar{a}')]_{<}^2$  there is the same decision about  $X_t \in \mathcal{R}_i$ .

First case.  $\forall t < r \in \text{FU}(\bar{a}'), (s \cup t \cup r, \bar{a}') \Vdash X_t \in \mathcal{R}_i$ . We take a partition  $\Pi$  so that any  $\text{dom}(a'_n)$  is entirely contained in an interval of  $\Pi$  and such that any  $\Pi$ -interval contains one of the  $\text{dom}(a'_n)$ .

We let  $\hat{X}_n := \bigcap_{\{t \in \text{FU}(\bar{a}') : \max(t) \leq \max(a'_n)\}} X_t$ . Since  $\mathcal{R}_i$  is selective, there is  $Z \in \mathcal{R}_i$ ,  $Z$  meets every  $\Pi$ -interval at most once and such that  $\min(Z) > \max(s)$  and  $\forall n \in \mathbb{Z} Z \setminus (n+1) \subseteq \hat{X}_n$ .

We show:

$$(4.5) \quad (s, \bar{a}') \Vdash Z \cap \mu(i) \setminus \{\min(Z \cap \mu(i))\} \subseteq \underline{A}.$$

Then we are done since  $\mathcal{R}_i$  contains all cofinite sets.

Since  $Z$  meets every interval of  $\Pi$  at most once it meets also every  $\bar{a}'_n$  at most once. For  $j \in Z \cap \mu(i)$  let  $a'_{k(j)}$  the element of  $\bar{a}'$  such that  $j \in (a'_{k(j)})^{-1}[\{i\}]$ . Now let  $j_0 = \min(Z \cap \mu(i))$ . Let  $(t, \bar{b}) \geq (s, \bar{a}')$ , and  $(t, b) \Vdash j_0, x \in Z \cap \mu, j_0 < x$ . By the definition of  $Z$ , there is  $k(x) > k(j_0)$ ,  $x \in a'_{k(x)}$ . Also by the choice of  $Z$ ,  $Z \setminus (j_0 + 1) \subseteq \hat{X}_{j_0}$ . Now  $k(j_0) \leq j_0$  and hence  $Z \subseteq (j_0 + 1) \subseteq X_{a'_{k(j_0)}}$ . By the definition of the forcing order,  $t = s \cup a'_{i_0} \cup \dots \cup a'_{i_\ell}$  and  $k(j_0)$  and  $k(x)$  are among the  $i_r$ ,  $r = 1, \dots, \ell$ . So we have  $x \in \hat{X}_{k(j_0)}$  and hence  $x \in X_{a'_{i_0} \cup \dots \cup a'_{k(j_0)}}$  and hence by definition of  $X_{t'}$  for  $t' = a'_{i_0} \cup \dots \cup a'_{k(j_0)} \in \text{FU}(\bar{a}')$  and for  $r' = a'_{k(j_0)+1} \cup \dots \cup a'_{i_\ell}$  and the choice of  $\bar{a}'$ ,  $(t, \bar{b}) \Vdash x \in \underline{A}$  and (4.5) is proved.

Second Case:  $X_t \notin \mathcal{R}_i$  is this majority decision. Then  $Y_t = \omega \setminus X_t \in \mathcal{R}_i$ . By the neatness,

$$Y_t = \{i \in \omega : (\forall r \in \text{FU}(\bar{a} ; \text{past } t))((s \cup t \cup r, \bar{a} ; \text{past } r) \Vdash i \notin \underline{A})\}.$$

Now we go on with the proof as in the first case and get a set  $Z \in \mathcal{R}_i$  and  $\bar{a}' \in \mathcal{U}$ ,  $\bar{a}' \sqsubseteq \bar{a}$ , such that  $(s, \bar{a}') \Vdash Z \cap \mu(i) \setminus \{\min(Z \cap \mu(i))\} \subseteq \omega \setminus \underline{A}$ .

We now prove that  $\text{fil}(\mathcal{R}_i \cup \{\mu(i)\})$  is a selective ultrafilter. We are given  $\langle A_j : j \in \omega \rangle$  such that  $\Vdash "A_j \text{ decreases with } j \text{ and } A_j \in \text{fil}(\mathcal{R}_i \cup \{\mu(i)\})^{\tilde{\mathbb{M}}(\mathcal{U})}"$ . Let  $(s, \bar{a})$  be neat for  $\langle A_j : j < \omega \rangle$ . We produce  $Z \in \mathcal{R}_i$  and  $(s, \bar{a}') \geq (s, \bar{a})$  such that

$$(4.6) \quad (s, \bar{a}') \Vdash (\forall x \in Z \cap \mu(i))(Z \cap \mu(i) \setminus (x+1) \subseteq \underline{A}_x).$$

As in the previous subclaims we let

$$(4.7) \quad X_t := \{i \in \omega : (\forall r \in \text{FU}(\bar{a} ; \text{past } t)) (s \cup t \cup r, \bar{a} ; \text{past } r) \Vdash i \in A_{\max(t)}\}$$

By Taylor's theorem, applied to  $f: [\text{FU}(\bar{a})]_{<}^2 \rightarrow 2$  with  $f(t_1, t_2) = \text{yes/no}$  if  $(s \cup t_1 \cup t_2, \bar{a} ; \text{past } t_2) \Vdash X_{t_1} \in / \notin \mathcal{R}_i$  and the Milliken–Taylor ultrafilter

$\mathcal{U}$ , there is  $\bar{a}' \subseteq \bar{a}$ ,

$$\bar{a}' \in \mathcal{U} \wedge$$

$$(4.8) \quad (\forall r \text{ FU}(\bar{a}' ; \text{past } t))(s \cup t \cup r ; \bar{a}' ; \text{past } r) \text{ decides } X_t \in \mathcal{R}_i \wedge$$

the decision does not depend on  $r$ .

First case.  $\forall t < r \in \text{FU}(\bar{a}')$ ,  $(s \cup t \cup r, \bar{a}') \Vdash X_t \in \mathcal{R}$ . We let  $\Pi$  be a partition of  $\omega$  into finite intervals so that any  $\text{dom}(a'_n)$  is entirely contained in an interval of  $\Pi$  and such that any  $\Pi$ -interval contains one of the  $\text{dom}(a'_n)$ .

$$(4.9) \quad \hat{X}_n := \bigcap_{\{t \in \text{FU}(\bar{a}') : \max(t) \leq \max(a'_n)\}} X_t.$$

Since  $\mathcal{R}_i$  is selective, there is  $Z \in \mathcal{R}_i$ ,  $Z$  meets every  $\Pi$ -interval at most once and such that  $\min(Z) > \max(s)$  and

$$(4.10) \quad \forall n \in \mathbb{N} \quad Z \setminus (n+1) \subseteq \hat{X}_n.$$

Now we prove statement (4.6): Since  $Z$  meets every interval of  $\Pi$  at most once it meets also every  $(a'_n)^{-1}[\{i\}]$  at most once. For  $x \in Z \cap \mu(i)$  let  $a'_{k(x)}$  this unique element of  $\bar{a}'$ . Let  $(t, \bar{b}) \geq (s, \bar{a}')$ , and  $(t, b) \Vdash j, x \in Z \cap \mu(i), j < x$ . By the definition of  $Z \cap \mu(i)$ , there is  $k(x) > k(j)$ ,  $x \in a'_{k(x)}$ . Also by the choice of  $Z$ ,  $Z \setminus (j+1) \subseteq \hat{X}_j$ , and hence  $Z \setminus (j+1) \subseteq X_{a'_{k(j)}}$ . By the definition of the forcing order,  $t = s \cup a'_{i_0} \cup \dots \cup a'_{i_\ell}$  and  $k(j)$  and  $k(x)$  are among the  $i_r$ ,  $r = 1, \dots, \ell$ . So we have  $x \in \hat{X}_k$  and hence  $x \in X_{s \cup \dots \cup a'_{k(j)}}$  and hence by definition of  $X_{t'}$  for  $t' = a'_{i_0} \cup \dots \cup a'_{k(j)} \in \text{FU}(\bar{a}')$  and  $r' = a'_{k(j)+1} \cup \dots \cup a'_{i_\ell}$  and the choice of  $\bar{a}'$ ,  $(t, \bar{b}) \Vdash x \in A_j$  and (4.6) is proved.

Second case:  $X_t \notin \mathcal{R}_i$  is this majority decision. Then  $Y_t = \omega \setminus X_t \in \mathcal{R}_i$ . We let  $\hat{Y}_n := \bigcap_{\{t \in \text{FU}(\bar{a}') : \max(t) \leq \max(a'_n)\}} Y_t$  and since  $\mathcal{R}_i$  is selective there is  $Z \in \mathcal{R}_i$  such that  $(\forall z \in Z)(Z \setminus (z+1) \subseteq \hat{Y}_z)$ . Then

$$(4.11) \quad (s \cup a'_0, \bar{a}' ; \text{past } a'_0) \Vdash A_{\max(a'_0)} \cap Z \cap \mu = \emptyset.$$

Suppose, for a contradiction, that there is  $(s \cup a'_0 \cup r, \bar{b} ; \text{past } r) \geq (s \cup a'_0, \bar{a}' ; \text{past } a'_0)$  with  $(s \cup a'_0 \cup r, \bar{b} ; \text{past } r) \Vdash x \in A_{\max(a'_0)} \cap Z$  and  $x \in r^{-1}[\{i\}]$ . Then  $x \in Z \setminus (\max(a'_0)+1)$  and hence  $x \in \hat{Y}_{\max(a'_0)}$ . So  $(s \cup a'_0 \cup r, \bar{b} ; \text{past } r) \Vdash x \notin A_{\max(a'_0)}$ . This contradiction shows  $(s \cup a'_0, \bar{a}' ; \text{past } a'_0) \Vdash A_{\max(a'_0)} \cap Z \cap \mu = \emptyset$ . However, the latter contradicts the choice of  $\langle A_m : m \in \omega \rangle$ . So the case assumption for the second case are never fulfilled. Thus we have proved that  $(\mathcal{R}_i \cup \{\mu(i)\})^{+\mathbb{M}(\mathcal{U})}$  is a selective coideal.

Now we prove conclusions (1b) and (1c): Suppose that  $\Vdash h$  is a finite-to-one surjective weakly increasing function and  $h(\text{fil}(\mathcal{R}_i \cup \{\mu(i)\})) = h(\text{fil}(\mathcal{R}_j \cup \{\mu_j\}))$ . We let  $(s, \bar{a})$  be neat for  $h$ . For  $t \in \text{FU}(\bar{a})$ , we let

$$(4.12) \quad h_t = \{(j, y) : (\forall r \in \text{FU}(\bar{a} ; \text{past } t)) (s \cup t \cup r, \bar{a} ; \text{past } r) \Vdash h(j) = y)\}.$$

Let  $\Pi$  be an interval partition such that each  $\text{dom}(a_n)$  is entirely contained in an interval of  $\Pi$  and such that each interval of  $\Pi$  contains at least one  $\text{dom}(a_n)$ . Now  $h_t \in \mathbf{V}$  is finite-to-one. Hence there is  $X_{i,t} \in \mathcal{R}_i$  and  $X_{j,t} \in \mathcal{R}_j$  and  $E_t \in \mathcal{E}$  such that  $h_t[X_{j,t}] \cap h_t[X_{j,t}] = \emptyset$  and  $h_t[X_{i,t}] \cap h_t[E_t] = \emptyset$ . We let

$$(4.13) \quad \hat{X}_{i,n} = \bigcap_{t \in \text{FU}(\bar{a}), \max(t) \leq \max(a_n)} X_{i,t}$$

analogous for  $j$  and  $\hat{E}_n = \bigcap_{t \in \text{FU}(\bar{a}), \max(t) \leq \max(a_n)} E_t$ . There is  $E \in \mathcal{E}$  such that for infinitely many  $n$ ,  $E \setminus n \subseteq \hat{E}_n$  (see [11, Lemma 4.4]), say for  $n \in B$ . We let  $\Pi'$  be an interval partition that is gotten from  $\Pi$  by merging adjacent intervals so that each interval of  $\Pi'$  contains an element of  $B$ . Let  $j_0 = \min(B)$ .

We let  $Z_i \in \mathcal{R}_i$ ,  $Z_j \in \mathcal{R}_j$  such that  $\forall n \in Z_i (Z_i \setminus (n+1)) \subseteq \hat{X}_{i,n}$  and the analogon for  $j$  holds and such that  $Z_i$  meets each interval of  $\Pi'$  at most once and such that  $Z_i$  and  $Z_j$  do not meet the same  $\Pi'$ -interval nor adjacent intervals. We let  $\bar{a}' \sqsubseteq \bar{a}$  be such that  $\bar{a}' \in \mathcal{U}$ , and such that each interval of  $\Pi'$  contains just one member of  $\bar{a}'$  and this member lies entirely in the interval. (So here we do not only thin out but also use the merging process allowed in  $\sqsubseteq$  if more than one block of  $\bar{a}$  lies in the same  $\Pi'$ -interval.)

Then we let  $(Z_i \cap \mu(i))^* = (Z_i \cap \mu(i)) \setminus \{\min(Z_i \cap \mu(i))\}$  and have

$$(4.14) \quad (s, \bar{a}') \Vdash \underline{h}[(Z_i \cap \mu(i))^*] \cap \underline{h}[(Z_j \cap \mu(j))^*] = \emptyset$$

and

$$(4.15) \quad (s, \bar{a}') \Vdash \underline{h}[(Z_i \cap \mu(i))^*] \cap \underline{h}[(E \cap [j_0, \infty))^*] = \emptyset.$$

The technique of proof for the Equations (4.14) (4.15) is the same technique we already saw twice, in the proof of (4.5) and (4.6): For  $x \in Z_i \cap \mu(i)$  let  $a'_{k(x,i)}$  this unique element of  $\bar{a}'$  such that  $x \in (a'_{k(x,i)})^{-1}[\{i\}]$ . Let  $(t, \bar{b}) \geq (s, \bar{a}')$ , and  $(t, \bar{b}) \Vdash x(i), y(i) \in Z_i \cap \mu(i), x(i) < y(i) \wedge x(j), y(j) \in Z_j \cap \mu(j), x(j) < y(j)$ . By the definition of  $Z_i$ , there is  $k(y, i) > k(x, i)$ ,  $x \in (a'_{k(x,i)})^{-1}[\{i\}]$ . Also by the choice of  $Z_i$ ,  $Z_i \setminus (x(i)+1) \subseteq \hat{X}_{x(i)}$  and hence  $Z_i \setminus (x(i)+1) \subseteq X_{a'_{k(x,i)}}$ . By the definition of the forcing order,  $t = s \cup a'_{i_0} \cup \dots \cup a'_{i_\ell}$  and  $k(x, i), k(y, i), k(x, j), k(y, j)$  are among the  $i_r$ ,  $r = 1, \dots, \ell$ . So we have  $x \in \hat{X}_{\max(k(x,i), k(x,j))}$  and hence  $x \in X_{i, a'_{i_0} \cup \dots \cup a'_{k(j)}}$  and hence by definition of  $X_{i, t'}$  for  $t' = a'_{i_0} \cup \dots \cup a'_{k(x,i)} \in \text{FU}(\bar{a}')$  and  $r' = a'_{k(x,i)+1} \cup \dots \cup a'_{i_\ell}$  and the choice of  $\bar{a}'$ ,  $(t, \bar{b}) \Vdash h(y(i)) = h_{t'}(y(i)) \wedge y(i) \in X_{i, t'}$ . By definition of  $X_{j, t'}$  for  $t' = a'_{i_0} \cup \dots \cup a'_{k(x,j)} \in \text{FU}(\bar{a}')$  and  $r' = a'_{k(x,j)+1} \cup \dots \cup a'_{i_\ell}$  and the choice of  $\bar{a}'$ ,  $(t, \bar{b}) \Vdash h(y(j)) = h_{t'}(y(j)) \wedge y(j) \in X_{j, t'}$ . So  $(t, \bar{b}) \Vdash h_t(y(i)) \neq h_t(y(j))$  and (4.14) is proved. For (4.15) we use that  $B$  is thick enough so that  $Z_i$  is a diagonalisation that makes the argument work.

The  $P$ -point  $\mathcal{E}$  is preserved by Eisworth's Theorem 3.5. The very last sentence of conclusion (1c) is proved in [23, Theorem 4.5]. The theorem



stated there, is slightly too weak for our purpose. However, the proof there uses only that  $\mathcal{E}$  is a  $P$ -point, whereas  $\mathcal{W}$  can be arbitrary.  $\square$

Now we define the relations  $\bar{R}$  for which we want to preserve statements of the form  $\forall f \exists g f R g$ . Also they will be defined inductively, by referring in  $R_{n,\alpha}$  to the  $\mathbb{P}_\gamma$ -names  $\mathcal{R}_{i,\gamma}$ ,  $\gamma < \alpha$ . So  $R_{n,\alpha}$  is a relation in  $\mathbf{V}_\alpha$ . Both inductions are carried on simultaneously. We let  $\bar{R} = \langle R_{n,\alpha} : \alpha \leq \omega_2, n \in \omega \rangle$ ,  $\alpha$  being the stage,  $n$  the size of the "mistake" in properties of the kind "for all but finitely many". In each step limit step  $\alpha$ , we define  $R_{n,\alpha}$  and  $\bar{g}_{\alpha,a}$  for  $a \in [\omega_1]^\omega$  as in Lemma 4.14, show that  $\mathbb{P}_\alpha$  preserves  $R_{n,\alpha}$  and then then define (rather choose)  $\mathcal{R}_{i,\alpha}$ . Then we define a relation  $R_{n,\alpha+1}$  and a notion of  $(\bar{R}, \mathcal{S}, \bar{g})$ -preservation for forcing with  $\mathbb{P}_{\alpha+1}$ , then we define  $\bar{g}_{\alpha+1,a}$  and then verify that  $\mathbb{P}_{\alpha+1}$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -preserving and so on.

**Definition 4.10.** Assume that  $\langle \mathcal{R}_{i,\gamma} : i = 1, \dots, k, \gamma < \alpha \rangle$  is an ascending sequence of Ramsey ultrafilters  $\mathcal{R}_{i,\gamma} \in \mathbf{V}_\gamma$  and in  $\mathbf{V}_\gamma$ ,  $\mathcal{R}_{i,\gamma}$  is not nearly coherent to  $\mathcal{R}_{j,\gamma}$  and not nearly coherent to  $\mathcal{E}$ . We say  $f R_{n,\alpha} \bar{g}$  if the following holds in  $\mathbf{V}_\alpha$ :

- (1)  $f = (\bar{A}, h)$ , for  $n \in \omega$ ,  $f(n) = ((\chi_{A_{1,\ell}} \upharpoonright (n+1), \dots, \chi_{A_{k,\ell}} \upharpoonright (n+1))_{\ell \leq n}, h \upharpoonright n+1)$  for  $\chi_{A_{i,\ell}}$  being the characteristic function of  $A_{i,\ell}$ ,
- (2)  $\bar{A} = \langle \bar{A}_i : i = 1, \dots, k \rangle$ . For  $i = 1, \dots, k$ ,  $\bar{A}_i = \langle A_{i,\ell} : \ell \in \omega \rangle$  is a  $\subseteq$ -descending sequence such that

$$(\forall X \in \text{fil}(\bigcup_{\gamma < \alpha} (\mathcal{R}_{i,\gamma} \cup \{\mu_\gamma(i)\}))) (X \cap A_{i,\ell} \in [\omega]^\omega),$$

- (3)  $h$  is finite-to-one,
- (4)  $\bar{g} = \langle g_i : i = 1, \dots, k \rangle$  is a sequence of infinite subsets of  $\omega$ ,
- (5) For  $i = 1, \dots, k$ ,

$$\begin{aligned} & (\forall \ell \in g_i) (g_i \setminus (\ell + 1, n + 1) \subseteq A_{i,\ell}) \wedge \\ & (\forall X \in \bigcup_{\gamma < \alpha} (\mathcal{R}_{i,\gamma} \cup \{\mu_\gamma(i)\})) (X \cap g_i \in [\omega]^\omega) \wedge \\ & \bigwedge_{j \in \{1, \dots, k\} \setminus \{i\}} h[g_j] \cap h[g_i] \subseteq n \wedge \\ & \exists E \in \mathcal{E} h[g_i] \cap h[E] \subseteq n. \end{aligned}$$

The relations  $R_{n,\alpha}$  are too complicated for an iteration theorem in the style of [35, Ch. XVIII], since the quantifiers in (2) and in (5) range over sets of size  $\aleph_1$ . Hence we weaken the relations by weakening (2) and (5) as follows:

**Definition 4.11.** Under CH, wet  $N \prec H(\chi)$ ,  $N \cap \omega_1 = a$ . (Under CH,  $N \cap H(\omega_1)$  is determined by  $N \cap \omega_1$ , and our definition makes sense only under this condition.)

We say  $fR_{n,\alpha,a}\bar{g}$  if the modification of  $fR_{n,\alpha}\bar{g}$  by writing the quantifiers  $(\forall X \in N \cap \bigcup_{\gamma < \alpha} (\mathcal{R}_{i,\gamma} \cup \{\mu_\gamma(i)\}))$  and  $\exists E \in \mathcal{E} \cap N$  in items (2) and (5) holds.

This will be used only for  $f \in N$ .

We fix a stationary subset  $\mathcal{S} \subseteq [\omega_1]^\omega$ .

We write  $\bar{g}$  for  $\langle g_i : i = 1, \dots, k \rangle$  and  $\bar{g}_{\alpha,a} = \langle g_{i,\alpha,a} : i = 1, \dots, k \rangle$ .  $\bar{g}_{\alpha,a}$  are in general not element of  $N$ . The following is a modification of [35, Ch. XVIII, Def. 3.2] for the case that the "coverers"  $\bar{g}$  are reals from the extension. Usually for preserving e.g. unboundedness in  $\leq^*$ , the coverers are taken from the ground model.

**Definition 4.12.** (1) Let  $a := N \cap \omega_1 \in \mathcal{S}$  and suppose  $N \in \bigcup_{\gamma < \alpha} \mathbf{V}_\alpha$ .  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good at stage  $\alpha$  means that the following holds: There is a  $\mathbb{P}_\alpha$ -name  $\bar{g}_{\alpha,a}$  such that

$$\Vdash_{\mathbb{P}_\alpha} (\forall f \in N \cap \text{dom}(R_{0,n,\alpha})) (\exists n \in \omega) fR_{n,\alpha,a}\bar{g}_{\alpha,a}.$$

Note that this is a mixture of stages:  $\bar{g}$  is from stage  $\alpha$  and  $f \in N$ ,  $N \in \bigcup_{\gamma < \alpha} \mathbf{V}_\gamma$ .

- (2) We say  $(\bar{R}, \mathcal{S}, \bar{g})$  fully covers for stage  $\alpha$  iff  $\mathcal{R}_{i,\gamma}$ ,  $\gamma < \alpha$ , are defined for some  $x \in H(\chi)$ , for every countable  $N \prec \mathcal{H}(\chi)$ ,  $N \in \bigcup_{\gamma < \alpha} \mathbf{V}_\gamma$  to which  $(\bar{R}, \mathcal{S}, \bar{g})$  and  $x$  belong and which fulfils  $N \cap \omega_1 \in \mathcal{S}$  we have that  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good at stage  $\alpha$ .

In the following the letter  $k$  has nothing to do with the number of ultrafilters. Now a notion of preserving that can be carried onwards through a countable support iteration of suitable iterands is defined:

**Definition 4.13.** (Compare to [35, Ch. XVIII, Def. 3.4], here we take it in the easily iterable form [27, 4.5]) We say  $\mathbb{P} = \langle \mathbb{P}_\beta, \mathbb{M}(\mathcal{U}_\gamma) : \gamma < \alpha, \beta \leq \alpha \rangle$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -preserving iff the following holds for any  $\chi$ ,  $\chi_1$ ,  $N$ ,  $p \in \mathbb{P} \cap N$ ,  $k < \omega$ : Assume

- (\*) (i)  $\chi_1$  is large enough and  $\chi > 2^{\chi_1}$ ,
- (ii)  $N \prec \mathcal{H}(\chi)$  is countable,  $N \cap \bigcup \mathcal{S} = a \in \mathcal{S}$ , and  $\mathbb{Q}, \mathcal{S}, \bar{g}, \chi_1 \in N$ ,
- (iii)  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good at stage  $\alpha$  and  $p \in \mathbb{P} \cap N$ ,
- (iv)  $k \in \omega$  and for  $\ell < k$  we have a  $\mathbb{P}$ -name for a function  $\underline{f}_\ell \in N$ , and  $\Vdash_{\mathbb{P}} \underline{f}_\ell \in \text{dom}(R_{0,\alpha,a})$
- (v) for  $\ell < k$ ,  $m < \omega$ ,  $f_{m,\ell}^*$  is a function in  $N$ ,
- (vi) for  $n < \omega$ ,  $p \leq p_n \leq p_{n+1}$ ,
- (vii) for  $x \in \text{dom}(f_{m,\ell}^*)$ ,  $\ell < k$ , for every  $m$  there is  $n_0$  such that for  $n \geq n_0$ ,  $p_n \Vdash_{\mathbb{P}} \underline{f}_\ell(x) = f_{m,\ell}^*(x)$ ,
- (viii) we have for  $\ell < k$ ,  $m < \omega$ ,  $\mathbb{P} \Vdash f_{m,\ell}^* R_{\beta_\ell^m, \alpha, a} \bar{g}_{\alpha,a}$  for some  $\beta_\ell^m \in \omega$ ,  $\beta_\ell^{m+1} \leq \beta_\ell^m$ , and  $\beta_\ell^* = \lim_{m \rightarrow \omega} \beta_\ell^m$ ,

(ix) if  $\mathcal{I}$  is a dense open set of  $\mathbb{P}$  and  $\mathcal{I} \in N$ , then for some  $n$ ,  $p_n \in \mathcal{I}$ .

Then there is  $q \geq p$ ,  $q \in \mathbb{P}$  that is  $(N, \mathbb{P})$ -generic and

- (a) for  $\ell < k$ ,  $q \Vdash_{\mathbb{P}} (\exists \gamma_\ell \in \omega, \gamma_\ell \leq \beta_\ell^*) (f_\ell R_{\gamma_\ell, \alpha, a} \bar{g}_{\alpha, a})$ ,
- (b)  $q \Vdash N[\mathbf{G}_{\mathbb{P}}]$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good for stage  $\alpha$  with the same witness  $\bar{g}_{\alpha, a}$ .

Note that  $\alpha$  is a parameter with a new meaning, the stage: The relations depend on the  $\mathcal{R}_{i, \gamma}$ ,  $\gamma < \alpha$ , in stage  $\alpha$ . So all is about names. The parameter  $\beta$  from [35, Chapter XVIII] corresponds to our  $n$ , the mistake. Conclusion (a) expresses a sort of directedness:  $\bar{g}_{\alpha, a}$  is the same for any  $f_\ell$ ,  $\ell < k$ .

We will use the possibility to work with unboundedly many  $k$  in the proof of the preservation of “ $(\bar{R}, \mathcal{S}, \bar{g})$ ” for iterations when the cofinality of the iteration length is countable.

Here is the use of the relations: Lemma 4.14, the Successor Theorem 4.5, the next two theorems and their respective lemmata and Lemma 4.19, Prop. 4.20 are proved together by induction on  $\alpha$ .

**Lemma 4.14.** *Suppose that  $\mathbb{P}_\gamma$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -preserving for stage  $\gamma$  for  $\gamma < \alpha$ , and  $\mathcal{R}_{i, \gamma}$ ,  $i = 1, \dots, k$ , are increasing with  $\gamma$  pairwise non nearly coherent ultrafilters (constructed according to Prop. 4.20) Then there for every  $N \in \bigcup_{\gamma < \alpha} \mathbf{V}_\gamma$  with  $\mathbb{P}_\alpha \in N$  there is  $\bar{g}_{\alpha, a}$  such that  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good at stage  $\alpha$ . So we have that  $(\bar{R}, \mathcal{S}, \bar{g})$  fully covers at stage  $\alpha$ .*

*Proof.* We first consider  $\alpha$  being a limit. Let  $f_m^*$  be an enumeration of all descending sequences in  $\bigcup_{\gamma < \alpha, \gamma \in a} \text{fil}(\mathcal{R}_{i, \gamma} \cup \{\mu_\gamma(i)\})^\alpha$  that are in  $\mathbf{V}_\gamma \cap N$  for some  $\gamma < \alpha$ ,  $\gamma \in a$ , combined with  $h_m \in \mathbf{V}_\gamma \cap N$ , enumerating all finite to one functions in  $\mathbf{V}_\gamma \cap N$  for  $\gamma < \alpha$ . Note here the filter is meant, not the positive sets. We can fulfil Def 4.13 (\*) (vii) with just these. We define  $g_{\alpha, a}$  be induction: Let  $f_m^*$  have the  $A$ -component  $\langle \langle A_{i, \ell}^m : \ell \in \omega \rangle : i = 1, \dots, k \rangle$  and the  $h$ -component  $h_m$ .

$$\begin{aligned} x_{i, 0} &= 0, \\ x_{i, n+1} &\in \bigcap \{A_{i, \ell}^m : m \leq n, \ell \leq x_{i, n}\} \\ &\quad \text{such that } (\forall n \leq m)(h_m(x_{i, n+1}) \neq h_m(x_{j, n+1})) \text{ and} \\ &\quad \text{for some } E^n \in \mathcal{E}, (\forall m \leq n)(h_m(x_{i, n+1}) \notin h_m[E^n]). \end{aligned}$$

Then we take  $B$  for  $E^n$ ,  $n \in \omega$  as in the proof of Theorem 4.5. Now

$$\bar{g}_{\alpha, a} = \langle \{x_{i, n} : n \in B\} : i = 1, \dots, k \rangle$$

shows that  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good at stage  $\alpha$ .

Now we consider  $\alpha = \gamma + 1$ . Let  $f_m^*$  be an enumeration of all descending sequences in  $\bigcup_{\gamma < \alpha, \gamma \in a} \mathcal{R}_{i, \gamma} \cup \{\mu_\gamma(i)\}$  that are in  $\mathbf{V}_\gamma \cap N$  combined with  $h_m \in N$ , enumerating all finite to one functions in  $\mathbf{V}_\gamma \cap N$ . We define  $g_{\alpha, a}$  be induction: Let  $f_m^*$  have the  $A$ -component  $\langle \langle A_{i, \ell}^m : \ell \in \omega \rangle : i = 1, \dots, k \rangle$

and the  $h$ -component  $h_m$ .

$$\begin{aligned} x_{i,0} &= \min(\mu_\gamma(i)), \\ x_{i,n+1} &\in \bigcap \{A_{i,\ell}^m : m \leq n, \ell \leq x_{i,n}\} \cap \mu_\gamma(i) \\ &\text{such that } h_m(x_{i,n+1}) \neq h_m(x_{j,n+1}) \text{ and} \\ &h_n(x_{i,n+1}) \notin h_n[E^n] \text{ for some } E^n \in \mathcal{E} \end{aligned}$$

Then we take  $B$  for  $E^n$ ,  $n \in \omega$  as in the proof of Theorem 4.5. Now

$$\bar{g}_{\alpha,a} = \langle \{x_{i,n} : n \in B\} : i = 1, \dots, k \rangle$$

shows that  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good at stage  $\alpha$ .  $\square$

The following theorem is the successor step of the induction step for  $\mathbb{P}_\alpha$  is  $(\bar{R}, \mathcal{S}, \bar{g})$  preserving.

**Theorem 4.15.** *The successor step. Let  $\mathcal{E}$  be a  $P$ -point. Let for  $i = 1, \dots, k$   $\mathcal{R}_i$  be a selective ultrafilter, such that  $\mathcal{R}_i$  is not isomorphic to  $\mathcal{R}_j$ , and  $\mathcal{R}_i$  not nearly coherent to  $\mathcal{E}$ . In addition let  $\mathcal{U}$  is a Milliken–Taylor ultrafilter over  $\mathbb{F}^k$ , such that  $\Phi(\mathcal{U}(i)) \subseteq \mathcal{R}_i$  and  $\Phi(\mathcal{U}) \not\leq_{\text{RB}} \mathcal{E}$ . Then*

- (1) *for any  $\mathbb{M}(\mathcal{U})$ -name  $\check{f}$  there are  $f_m^*, p_n$ , such that  $(\check{f}, (f_m^*)_m, (p_n)_n, \bar{g}_{1,a})$  fulfils  $(*)$  of Def. 4.13, and*
- (2)  *$\mathbb{M}(\mathcal{U})$  is  $(\mathcal{S}, \bar{R}, \bar{g})$ -preserving.*

*Proof.* In the proof of Theorem 4.5 we showed any  $\bar{g}_{1,a}$  that witnesses that  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good at stage 1, fulfils for any  $\mathbb{M}(\mathcal{U})$ -name  $\check{f} = (\bar{A}, \bar{h}) \in N$  up to a finite mistake all the criteria put on the  $Z_i \cap \mu(i)$  in either part of the proof. So together with  $f_m^*$  that is read off the  $X_t$ ,  $h_t$  from the proof of 4.5, it fulfils  $(*)$  of the definition of  $\mathbb{M}(\mathcal{U})$  is preserving. For  $q \geq p$  we take a diagonal intersection of all the  $\bar{a}'$  used in the proof of (2). Since  $\mathbb{M}(\mathcal{U})$  has the c.c.c., any condition is  $(N, \mathbb{M}(\mathcal{U}))$ -generic. Hence  $q$  is  $M(\mathcal{U})$ -generic. Hence  $q$  and  $\bar{g}_{\alpha,a}$  fulfil conclusions (a) and (b) of Def 4.17.  $\square$

Now we iterate the property " $\mathbb{Q}$  is  $(R, \mathcal{S}, \bar{g})$ -preserving". The first is the composition of two preserving forcings:

**Lemma 4.16.** ([35, Ch. XVIII, Claim 3.5])

- 1) *If  $(\bar{R}, \mathcal{S}, \bar{g})$  covers in  $\mathbf{V}_\alpha$ ,  $\mathbb{P}_\alpha$  is  $(R, \mathcal{S}, g)$ -preserving and  $\mathcal{R}_{i,\alpha}$ ,  $i = 1, \dots, k$  are chosen according to the induction hypothesis, and  $\mathbb{Q} = \mathbb{P}_{\alpha, \alpha+\beta}$  is an  $(\bar{R}, \mathcal{S}, \bar{g})$ -preserving forcing notion of length  $\beta$  in  $\mathbf{V}_\alpha$  (where  $\bar{R}_{[\alpha,\beta]}$  is defined with  $\mathcal{R}_{i,\alpha+\gamma}$ ,  $\gamma < \beta$ ). Then in  $\mathbf{V}^\mathbb{Q}$ ,  $(\bar{R}, \mathcal{S}, \bar{g})$  covers at stage  $\alpha + \beta$ .*
- 2) *The property " $(\bar{R}, \mathcal{S}, \bar{g})$ -preserving" is preserved by composition of forcing notions.*

*Proof.* (1) Let  $\mathbf{G}$  be  $\mathbb{P}_{\alpha+\beta}$ -generic over  $\mathbf{V}$ .  $N[\mathbf{G}] \prec \mathcal{H}(\chi)^{\mathbf{V}[\mathbf{G}]}$  for  $N$  being  $(\bar{R}, \mathcal{S}, \bar{g})$ -good in  $\mathbf{V}$  is a witness for covering in  $\mathbf{V}[\mathbf{G}]$ . We can take  $k = 0$  in  $(*)$ , so  $(*)$  is vacuously true and we get conclusion (b) of Def. 4.13.

(2) We fix  $\mathbb{P}_{\alpha+\beta} = \mathbb{P}_\alpha * \mathbb{P}_{[\alpha, \alpha+\beta)}$ , and write  $\mathbb{Q}_0 = \mathbb{P}_\alpha$ ,  $\mathbb{Q}_1 = \mathbb{P}_{[\alpha, \beta)}$ ,  $\chi$ ,  $\chi_1$ ,  $N$ ,  $a$ ,  $k$ ,  $f_\ell$ ,  $\beta_\ell$ ,  $f_{m,\ell}^*$  for  $\ell < k$ ,  $m < \omega$ ,  $p = (q_0^0, q_1^0)$ ,  $p^n = p_n = (q_0^n, q_1^n)$   $\bar{g}_{\alpha+\beta, a}$  as in (\*) of Definition 4.13 for  $\mathbb{P}_\alpha * \mathbb{P}_{[\alpha, \alpha+\beta)}$ . The subscript 0 stands for the  $\mathbb{P}_\alpha$ -part and the subscript 1 stand for the  $\mathbb{P}_{[\alpha, \alpha+\beta)}$ -part of a condition. We take  $p^0 = p$ . By condition (vi) of (\*) for each  $n < m < \omega$ ,  $q_0^m \Vdash_{\mathbb{Q}_0} q_1^0 \leq_{\mathbb{Q}_1} q_1^n \leq_{\mathbb{Q}_1} q_1^m$  hence without loss of generality by clause (ix) of (\*) by taking different names  $q_1^n$  that are above  $q_0^n$  the same,

(\*)<sub>1</sub>  $\Vdash_{\mathbb{Q}_0} q_1^0 \leq_{\mathbb{Q}_1} q_1^n \leq_{\mathbb{Q}_1} q_1^m$ , and

(\*)<sub>2</sub> for every  $x \in \omega$  or every sufficiently large  $n < \omega$ ,  $(\emptyset, q_1^n)$  forces  $f_\ell(x)$  to be equal to some specific  $\mathbb{Q}_0$ -name  $f'_{n,\ell}(x) \in N$  for each  $\ell < k$ .

Since  $\mathbb{Q}_0$  is  $(\bar{R}, \mathcal{S}, \bar{g})$  preserving there is  $q_0 \in \mathbb{Q}_0$  which is  $(N, \mathbb{Q}_0)$ -generic and is above  $q_0^0$  in  $\mathbb{Q}_0$  and forces  $N[\mathbf{G}_{\mathbb{Q}_0}]$  to be  $(\bar{R}, \mathcal{S}, \bar{g})$ -good and for some  $\gamma_\ell^* \leq \beta_\ell^*$ , we have  $(q_0, 0_{\mathbb{Q}_1}) \Vdash_{\mathbb{Q}_0 * \mathbb{Q}_1} \bigwedge_{\ell < k} f_\ell[\mathbf{G}_0] R_{\gamma_\ell^*, \alpha, a} g_{\alpha+\beta, a}$ . Let  $\mathbf{G}_0 \subseteq \mathbb{Q}_0$  be generic over  $\mathbf{V}$  and  $q_0 \in \mathbf{G}_0$ . We want to apply Definition 4.13 with  $N[\mathbf{G}_0]$ ,  $q_1^0[\mathbf{G}_0]$ ,  $\langle q_1^n[\mathbf{G}_0] : n < \omega \rangle$ ,  $\langle f_\ell[\mathbf{G}_0] : \ell < k \rangle$ ,  $\langle f'_{n,\ell}[\mathbf{G}_0] : \ell < k, n < \omega \rangle$ ,  $\langle \gamma_\ell^* : \ell < k \rangle$ ,  $\mathbb{Q}_1[\mathbf{G}_0]$  there in (\*) and check that all the items are fulfilled.

Clause (i) follows from clause (i) for  $\mathbb{Q}_0 * \mathbb{Q}_1$ ,

clause (ii): as  $q_0$  is  $(N, \mathbb{Q}_0)$ -generic we have  $N[\mathbf{G}_0] \cap \bigcup \mathcal{S} = N \cap \bigcup \mathcal{S} \in \mathcal{S}$ ,

clause(iii) holds by the choice of  $q_0$  and by conclusion (b) in Definition 4.13 for  $\mathbb{Q}_0$ ,

clause (iv) follows from clause (iv) for  $\mathbb{Q}_0 * \mathbb{Q}_1$ ,

clause (v): if  $x \in \omega$  then there are  $\ell$  and a  $\mathbb{Q}_0$ -name  $\tau \in N$  such that  $\Vdash_{\mathbb{Q}_0} [q_\ell^1 \Vdash_{\mathbb{Q}_1} f'_{m,\ell}(x) = \tau \in \omega]$ , as the set of  $(r_0, r_1) \in \mathbb{Q}_0 * \mathbb{Q}_1$  such that  $r_0 \Vdash_{\mathbb{Q}_0} r_1 \Vdash_{\mathbb{Q}_1} f'_{m,\ell}(x) = \tau$  for some  $\mathbb{Q}_0$ -name  $\tau$  is a dense open subset of  $\mathbb{Q}_0 * \mathbb{Q}_1$  some  $(q_\ell^0, q_\ell^1)$  is in it and there is such a  $\tau$ , by properness w.l.o.g.  $\tau \in N$ . So  $f'_{m,\ell}[\mathbf{G}_0] = \tau[\mathbf{G}_0] \in \omega$ .

clause (vi) was ensured by our choice (\*<sub>1</sub>),

clause (vii) by the choice of  $f'_{m,\ell}$  and  $\langle q_1^n : n < \omega \rangle$ ,

clause (viii) by the choice of  $q_0$  and  $\gamma_\ell^*$ ,

clause (ix) follows from clause(ix) for  $\mathbb{Q}_0 * \mathbb{Q}_1$  and a density argument as in (v). In details: If  $N[\mathbf{G}_0] \models \mathcal{I} \subseteq \mathbb{Q}_1$  is dense and open, then since  $\mathcal{I} \in N[\mathbf{G}_0]$  for some  $\mathcal{I}' \in N$  we have  $\Vdash_{\mathbb{Q}_0} \mathcal{I}'$  is a dense open subset of  $\mathbb{Q}_1$  and  $\mathcal{I}'[\mathbf{G}_0] = \mathcal{I}$ . Let  $\mathcal{J} = \{(r_0, r_1) \in \mathbb{Q}_0 * \mathbb{Q}_1 : r_1 \in \mathcal{I}'\}$ .  $\mathcal{J} \in N$  is a dense open subset of  $\mathbb{Q}_0 * \mathbb{Q}_1$ . Hence for every sufficiently large  $\ell$ ,  $(q_\ell^0, q_\ell^1) \in \mathcal{J}$  and so  $q_\ell^1[\mathbf{G}_0] \in \mathcal{I}'[\mathbf{G}_0] = \mathcal{I}$  and we finish.  $\square$

Now we head for the limit steps of countably cofinality. Here the situation is different, we cannot expect that just by adding one or countably many sets to the filter we get an ultrafilter again, as we have in ZFC:

**Proposition 4.17.** ([13, Prop. 1.4]) *Let  $\mathcal{F}_k$ ,  $k \in \omega$ , be a strictly increasing chain of filters. Then  $\bigcap_k \mathcal{F}_k$  is not ultra.*

Compare with [35, Ch. XVIII, Theorem 3.6] and to the version [27]:

**Theorem 4.18.** *The limit theorem. Suppose CH and  $\alpha \leq \aleph_2$ . Let  $\alpha$  be a limit ordinal and let  $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\mathcal{U}_\gamma) : \gamma < \alpha, \beta \leq \alpha \rangle$  be the countable support iteration of proper iterands. Let  $\mathcal{R}_{i,\gamma}$ ,  $i = 1, \dots, k$ , be pairwise non-nearly coherent selective ultrafilters in  $\mathbf{V}_\gamma$ , increasing in  $\gamma$ . Let  $\mathcal{E}$  be a  $P$ -point in  $\mathbf{V}_{\alpha_0}$ , not nearly coherent to  $\mathcal{R}_{i,\gamma}$  in  $\mathbf{V}_\gamma$ . If each  $\mathbb{M}(\mathcal{U}_\gamma)$  is  $(\mathcal{S}, \bar{R}, \bar{g})$ -preserving, then*

- (1)  $(\bar{R}, \mathcal{S}, \bar{g})$  covers at stage  $\alpha$ , and
- (2) for any  $\mathbb{P}_\alpha$ -name  $f$  there are  $f_m^*$ ,  $p_n$ , such that  $(f, (f_m^*)_m, (p_n)_n, \bar{g}_{\alpha,a})$  fulfils  $(*)$  of Def. 4.13 for  $\mathbb{P}_\alpha$ , and
- (3)  $\mathbb{P}_\alpha$  is  $(\mathcal{S}, \bar{R}, g)$ -preserving.

*Proof.* The covering part was proved in Lemma 4.14. We assume that all the  $g_{\alpha,a}$  appearing below are as there, hence fulfilling  $(*)$  of Def. 4.13. We prove by induction of  $\zeta \leq \alpha$  that for every  $\xi \leq \zeta$ , statement (2) for  $\mathbb{P}_\zeta/\mathbb{P}_\xi$  and that  $\mathbb{P}_\zeta/\mathbb{P}_\xi$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -preserving in  $\mathbf{V}^{\mathbb{P}_\xi}$ , moreover in Definition 4.13 we can get  $\text{dom}(q) \setminus \xi = \zeta \cap N$ . For  $\zeta = 0$  there is nothing to prove, for  $\zeta$  successor we use the previous lemma. So let  $\zeta$  be a limit. We first consider  $\text{cf}(\zeta) = \omega$ . We fix a strictly increasing sequence  $\langle \zeta_\ell : \ell < \omega \rangle$  with  $\zeta_0 = \xi$  and  $\sup \zeta_\ell = \zeta$ .

We let  $\{\tau_j : j \in \omega\}$  list the  $\mathbb{P}_\zeta$ -names of ordinals which belong to  $N$ . Let  $N \in bV_\xi$  be  $(\bar{R}, \mathcal{S}, \bar{g})$ -good for stage  $\zeta$ . In the following we use the convention that the first index indicates that we deal with a  $\mathbb{P}_{\zeta_\ell}$ -name  $\tau$  or  $f$  (for a  $\mathbb{P}_\zeta/\mathbb{P}_{\zeta_\ell}$ -name) and the second index is for the enumeration of the particular subset of  $N$ .

We choose by induction on  $j$ ,  $k_j < \omega$  such that

- (A)  $k_j < k_{j+1}$ ,
- (B) there is a sequence  $\langle \tau_{\ell,j} : \ell < j \rangle$  such that  $\tau_{\ell,j}$  is a  $\mathbb{P}_{\zeta_\ell}$ -name and
  - ( $\alpha$ )  $p_{k_j} \upharpoonright [\zeta_j, \zeta) \Vdash_{\mathbb{P}_\zeta} \tau_j = \tau_{j,j}$ ,
  - ( $\beta$ ) for  $\ell < j$  we have  $p_{k_j} \upharpoonright [\zeta_\ell, \zeta_{\ell+1}) \Vdash_{\mathbb{P}_{\zeta_{\ell+1}}} \tau_{\ell+1,j} = \tau_{\ell,j}$ ,
- (C) if  $j = i + 1$ ,  $\ell < i$  then  $\Vdash_{\mathbb{P}_{\zeta_{\ell+1}}} p_{k_i} \upharpoonright [\zeta_\ell, \zeta_{\ell+1}) \leq p_{k_j} \upharpoonright [\zeta_\ell, \zeta_{\ell+1})$ ,
- (D) if  $j = i + 1$  then  $\Vdash_{\mathbb{P}_\zeta} p_{k_i} \upharpoonright [\zeta_i, \zeta) \leq p_{k_j} \upharpoonright [\zeta_i, \zeta)$ .

Given  $k_i$ ,  $\langle \tau_{\ell,i} : \ell < i \rangle$  we by induction hypothesis the  $p$  that fulfil the requirement for  $p_{k_{i+1}}$  are dense in  $\mathbb{Q} \cap N$ , hence by (ix) there is a  $k_{i+1}$  such that  $p_{k_{i+1}}$  is in that dense set.

Now let  $f_\ell$ ,  $\ell < k$ , be given as in  $(*)$  of Def. 4.13. Let  $\{f_j : \ell < j < \omega\}$  list the  $\mathbb{P}_\zeta$ -names of members on  $N$  that are in  $\text{dom}(R_{0,\alpha,a})$ . For  $\ell < k$  let them be the  $f_{m,\ell}^*$  as given in  $(*)$  of Definition 4.13. Since  $N$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good,

$p_n$  from above can serve as  $p_n$  in (\*). We will now show how to choose  $f_{m,j}^* \in N$ ,  $m < \omega$ ,  $j < \omega$ .

Let  $h(j, x) < \omega$  be such that  $\tau_{h(j,x)} = f_j(x)$ . We can now define for  $n < \omega$ ,  $j < \omega$ ,  $f_{n,j}^*$  a  $\mathbb{P}_{\zeta_n}$ -name of a function from  $a$  to  $a$ . Let  $f_{n,j}^*(x) = \tau_{n,h(j,x)}$  if  $h(j, x) \geq n$  and  $\tau_{h(j,x),h(j,x)}$  if  $h(j, x) < n$ . So  $f_{0,j}^*(x) = f_j(x)$  for  $j < \omega$ . So also for the names  $f_{m,j}^*(x)$  we have (viii) of the hypothesis (\*), since (viii) holds objects  $f_{m,j}$  from there and the sets

$$\{f \in {}^\omega \omega : f R_{\beta,\zeta,a} g_{\zeta,a}^-\}$$

are closed sets for  $\beta \in \omega$ . (The quantifier  $\exists E \in \mathcal{E}_0 \cap N$  is harmless since there is always a pseudointersection of the countably many instances and we can repeat the work with a set  $B$  as in the proof of the successor theorem.)

We choose by induction on  $n$ ,  $q_n$ ,  $\alpha_{\zeta_n}^n$  for  $\ell < k + n$  such that

- (a)  $q_n \in \mathbb{P}_{\zeta_n}$ ,  $\text{dom}(q_n) \setminus \xi = N \cap \zeta_n$ ,  $q_{n+1} \restriction \zeta_n = q_n$ ,
- (b)  $q_n$  is  $(N, \mathbb{P}_{\zeta_n})$ -generic,
- (c)  $q_n \Vdash_{\mathbb{P}_{\zeta_n}} N[\mathbf{G}_{\mathbb{P}_{\zeta_n}}]$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good,
- (d)  $p_0 \restriction \zeta_0 \leq q_0$  in  $\mathbb{P}_{\zeta_0}$ ,
- (e)  $q_{n+1} \restriction \zeta_n \Vdash_{\mathbb{P}_{\zeta_n}} p_n \restriction [\zeta_n, \zeta_{n+1}) \leq p_{n+1} \restriction [\zeta_n, \zeta_{n+1})$  (in  $\mathbb{P}_{\zeta_{n+1}}/\mathbb{P}_{\zeta_n}$ ),
- (f) for  $\ell < k + n$ ,  $\alpha_{\zeta_n}^n$  is a  $\mathbb{P}_{\zeta_n}$ -name of an ordinal in  $a$ ,  $q_{n+1} \Vdash \alpha_{\zeta_n}^{n+1} \leq \alpha_{\zeta_n}^n$ ,  $\alpha_{\zeta_n}^0 \leq \beta_{\zeta_n}^*$ , for  $\ell < k$ ,
- (g) for  $\ell < k + n$ ,  $q_n \Vdash_{\mathbb{P}_{\zeta_n}} f_{n,\ell}^* R_{\alpha_{\zeta_n}^n, \zeta, a} \bar{g}_{\zeta, a}$ .

The induction step is by the induction hypothesis and by Definition 4.13 with  $k + n$  in the role of  $k$ . In the end we let  $q = \bigcup_{n < \omega} q_n$ .

We show that  $q$  is  $(N, \mathbb{P}_{\zeta})$ -generic and that it satisfies conclusions (a) and (b) of Def. 4.13. Let  $q \in \mathbf{G}_{\mathbb{P}_{\zeta}} \subseteq \mathbb{P}_{\zeta}$ ,  $\mathbf{G}_{\mathbb{P}_{\zeta}}$  be  $\mathbb{P}_{\zeta}$ -generic over  $\mathbf{V}$ .  $\mathbf{G}_{\mathbb{P}_{\xi}} = \mathbf{G}_{\mathbb{P}_{\zeta}} \cap \mathbb{P}_{\xi}$  for  $\xi < \zeta$  and  $\mathbf{G}_{\mathbb{P}_{\zeta_n}} = \mathbf{G}_{\mathbb{P}_{\zeta}} \cap \mathbb{P}_{\zeta_n}$ . Now for each  $\mathbb{P}_{\zeta}$ -name  $\tau$  for an ordinal there is some  $j$  such that  $\tau = \tau_j$ .  $q_j$  forces  $\tau_{j,j} \in N$  and  $p_j \restriction [\zeta_j, \zeta)$  forces  $\tau_j = \tau_{j,j}$ .  $p_j \restriction [\zeta_j, \zeta) \leq q$  by (d) and (e). So  $q$  forces  $\tau_j = \tau_{j,j}$  and  $q \Vdash \tau_j \in N \cap \text{On}$ , so  $q$  is  $(N, \mathbb{P}_{\zeta})$ -generic.

For each  $\ell$ ,  $\langle \alpha_{\zeta_n}^n : \ell \leq n < \omega \rangle$  is not increasing by (f) and hence eventually constant, say with value  $\alpha_{\zeta_n}^*$ . If  $x \in \omega$ ,  $j < \omega$ , then for  $n > h(j, x)$ ,  $p_n \Vdash f_j(x) = f_{n,j}^*(x)$ . So for every finite  $b \subseteq a$ ,  $\langle (f_{n,j}^* \restriction b)[\mathbf{G}_{\mathbb{P}_{\zeta_n}}] : n < \omega \rangle$  is eventually constant, equal to  $(f_j \restriction b)[\mathbf{G}_{\mathbb{P}_{\zeta}}]$ . By (g), for sufficiently large  $n$ ,

- (1)  $q \Vdash_{\mathbb{P}_{\zeta}} (f_j \restriction b)[\mathbf{G}_{\mathbb{P}_{\zeta}}] = (f_{n,j}^* \restriction b)[\mathbf{G}_{\mathbb{P}_{\zeta_n}}]$  and
- (2)  $q_n \Vdash f_{n,j}^*[\mathbf{G}_{\mathbb{P}_{\zeta_n}}] R_{\alpha_{\zeta_n}^n, \zeta, a} \bar{g}_{\zeta, a}$  and
- (3)  $\alpha_{\zeta_n}^n = \alpha_{\zeta_n}^*$ .



Since  $R_{\alpha_j^*, \zeta, a}$  is closed, and  $\langle f_{n,j}^* \restriction b[\mathbf{G}_{\mathbb{P}_\zeta}] : b \subseteq \omega, b \text{ finite} \rangle$  converges with increasing  $b$  and  $n$  to  $f_j$ , we get  $q \Vdash_{\mathbb{P}_\zeta} f_j R_{\alpha_j^*, \zeta, a} \bar{g}_{\zeta, a}$ . This finishes the proof of (b), that  $q \Vdash_{\mathbb{P}_\zeta} N[\mathbf{G}_{\mathbb{P}_\zeta}]$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -good. Now for (a) note that for there is  $n$  such that for  $\ell < k$ ,  $q_n \Vdash \alpha_\ell^* \leq \alpha_\ell^n[\mathbf{G}_{\mathbb{P}_\zeta}] \leq \beta_\ell^*$ . Thus we finished the proof for the limit of countable cofinality.

In the case of a limit of uncountable cofinality, all the  $\mathbb{P}_\zeta$ -names  $f \in N$  for functions in  $\text{dom}(\bar{R})$  and the  $f_\ell$ ,  $\ell < k$ , are  $\mathbb{P}_\xi$ -names for a  $\xi < \zeta$ . So item (1) is proved.  $\square_{4.18}$

The following shows how stationarity and properness allow to climb up from  $R_{0, \alpha, a}$  for stationary many  $a$  to  $R_{0, \alpha}$ :

**Lemma 4.19.** *Suppose that  $(\bar{R}, \mathcal{S}, \bar{g})$  fully covers at stage  $\alpha$  and that  $\mathbb{P}_\alpha$  is  $(\bar{R}, \mathcal{S}, \bar{g})$ -preserving. Given  $p \in \mathbb{P}_\alpha$  and  $\mathbb{P}_\alpha$ -names  $\tilde{f} = (\langle A_{i, \ell} : \ell \in \omega \rangle, \tilde{h})$  such that*

$$(4.16) \quad \begin{aligned} p \Vdash_{\mathbb{P}} & \text{ “ } \bigwedge_{i=1}^k (\langle A_{i, \ell} : \ell \in \omega \rangle \text{ is } \subseteq\text{-descending and } A_{i, \ell} \subseteq \omega \\ & \text{ and } (\forall X \in \bigcup_{\gamma < \alpha} \mathcal{R}_\gamma) X \cap A_{i, \ell} \in [\omega]^\omega), \text{ and} \\ & \tilde{h} \text{ finite-to-one).} \end{aligned}$$

Then there is a  $q \geq p$  and a  $\mathbb{P}_\alpha$ -name  $\bar{g} = (g_1, \dots, g_k)$  such that

$$(4.17) \quad \begin{aligned} q \Vdash_{\mathbb{P}} & \bigwedge_{i=1}^k \left( (\forall \ell \in g_i) (g_i \setminus (\ell + 1) \subseteq A_{i, \ell}) \wedge \right. \\ & (\forall X \in \bigcup_{\gamma < \alpha} \mathcal{R}_{i, \gamma}) (X \cap g_i \in [\omega]^\omega) \wedge \\ & \bigwedge_{i \neq j} h[g_i] \cap h[g_j] = \emptyset \wedge \\ & \left. \exists E \in \mathcal{E} h[g_i] \cap h[E] = \emptyset \right). \end{aligned}$$

*Proof.* We drop the tildes. Given  $f, p$  we let

$$(4.18) \quad \begin{aligned} (g_i = \bar{g}_{f, p})_i = & \{ (n, q) : \exists N \prec H(\chi)(\mathbb{P}_\alpha, f, p \in N, p \leq q, \\ & q \text{ is } (N, \mathbb{P}_\alpha)\text{-generic as in conclusion} \\ & \text{(a) and (b) of the definition of preserving} \\ & \wedge q \Vdash (f R_{m, \alpha, N \cap \omega_1} \bar{g}_{\alpha, N \cap \omega_1} \\ & \wedge n \geq m \wedge n \in (\bar{g}_{\alpha, a})_i) \} \end{aligned}$$

Suppose that  $p \Vdash \neg f R_{0, \alpha} g$ ,  $f = (\bar{A}, h)$ . Then there are densely many  $q \geq p$  such that

(1) there is  $X$  such that  $q \Vdash X \in \mathcal{R}_\gamma \wedge X \cap g_i = \emptyset$  or

- (2)  $q \Vdash g_i \setminus (n+1) \not\subseteq A_{i,n}$  or
- (3)  $q \Vdash \forall E \in \mathcal{E} \, \bar{h}[g_i] \cap \bar{h}[E] \in [\omega]^\omega$  or
- (4)  $q \Vdash \bar{h}[g_i] \cap \bar{h}[g_j] \in [\omega]^\omega$ .

There is no  $q$  with (2) or (3) or (4), by the definition of  $g_i$ . So we have that there are densely many  $q$  with (1). So there are club many  $N$ , such that densely many  $q \in N$ , with a respective witness  $X \in N$  and  $\gamma \in N$  or with  $E \in N$  force one instance of this contrary. Since  $\mathcal{S}$  is stationary, there is  $N$  in this club and  $a = N \cap \omega_1 \in \mathcal{S}$ . Now there is  $q \in N$  as in the conclusion of Def. 4.13. However,  $q$  forces that  $\bar{g}_{f,p} = \bar{g}_{\alpha,a}$ , and  $X \in N$ ,  $\gamma \in a$  and  $q \Vdash n \in (\bar{g}_{\alpha, N \cap \omega_1})_i \cap X$  for some  $n$ , contradiction. Note, also the  $X$  and the  $\mathcal{R}_{i,\gamma}$  are names. We use properness and genericity and the fact that we have equivalent hereditarily countable names for any real and that any  $N$  generic condition forces this equivalence, for a proof see, e.g., [22, Cor. 5.4]. So  $\bar{g}_{\alpha,a}$  based on knowledge from inside  $N$  (but of course constructed with an outside sequence covering the whole  $N$ ), is strong enough.  $\square$

The next step is Mathias's Proposition [21, Prop. 011] mentioned in the introduction: Now that we know  $\forall f \exists \bar{g} (f R_{0,\alpha} \bar{g})$ , we can complete in  $\mathbf{V}_\alpha$  the filter  $\text{fil}(\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma})^\alpha$  to a selective ultrafilter  $\mathcal{R}_{i,\alpha}$  again that has the additional properties concerning  $\mathcal{E}$  and the non-near-coherence of  $\mathcal{R}_{i,\alpha}$  and  $\mathcal{R}_{j,\alpha}$  for  $i \neq j$ . Whereas in the case of a successor  $\alpha = \gamma + 1$  the ultrafilter  $\mathcal{R}_{i,\alpha}$  is already given as  $\mathcal{R}_{i,\alpha} = \text{fil}(\mathcal{R}_{i,\gamma} \cup \{\mu_\gamma(i)\})^\alpha$ , in the limit case  $\alpha$  of uncountable cofinality we use the following construction to find a selective ultrafilter  $\mathcal{R}_{i,\alpha} \supseteq \bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma}$ :

**Proposition 4.20.** *Assume CH. Let  $\alpha < \omega_2$ ,  $\text{cf}(\alpha) = \omega$ . If  $\mathbb{P}_\alpha$  is  $(\mathcal{S}, \bar{R}, \bar{g})$ -preserving, then there are selective ultrafilters  $\mathcal{R}_{i,\alpha} \supseteq \text{fil}(\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma})^\alpha$  that are pairwise non nearly coherent and not nearly coherent to  $\mathcal{E}$ .*

*Proof.* We rework and strengthen Mathias' Prop. 0.11 from [21], in  $\mathbf{V}_\alpha$ . Let  $\langle \langle X_{i,\ell}^\xi : i = 1, \dots, k \rangle : \ell \in \omega \rangle, h^\xi : \xi < \omega_1 \rangle$  enumerate any pair consisting of descending sequence in  $([\omega]^\omega)^k$  and any finite-to-one function, such that each pair appears cofinally often. By induction on  $\zeta \in \omega_1$  we define a  $k$ -tuple of filters  $(\mathcal{F}_i^\zeta : i = 1, \dots, k)$  such that for  $\zeta < \omega_1$ :

- (1)  $\mathcal{F}_i^\zeta \supseteq \text{fil}(\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma})^\alpha$ .
- (2)  $\mathcal{F}_i^\zeta$  is countably generated over  $\text{fil}(\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma})^\alpha$  and  $\subseteq \text{fil}(\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma})^{+\alpha}$ .
- (3)  $(\exists n \in \omega)((\omega \setminus X_{i,n}^\zeta) \in \mathcal{F}_i^\zeta)$  or  $(\exists Y_i \in \mathcal{F}_i^{\zeta+1})(\forall n \in Y_i)((Y_i \setminus (n+1)) \subseteq X_{i,n}^\zeta) \wedge \bigwedge_{i \neq j, Y_i, Y_j \text{ defined}} (h^\zeta[Y_i] \cap h^\zeta[Y_j] = \emptyset) \wedge \bigwedge_{i, Y_i \text{ defined}} (\exists E \in \mathcal{E})(h^\zeta[Y_i] \cap h^\zeta[E] = \emptyset)$ .
- (4) For limit  $\zeta$  and also for  $\zeta = \omega_1$ ,  $\mathcal{F}_i^\zeta = \bigcup_{\delta < \zeta} \mathcal{F}_i^\delta$ .

The induction is routine. For a detailed proof, [21]. Item (3) is possible because of the relation  $\bar{R}_\alpha$  and Lemma 4.19. The  $\mathcal{F}_i^{\omega_1} =: \mathcal{R}_{i,\alpha}$  are pairwise

non-nearly coherent selective ultrafilters that are not nearly coherent to  $\mathcal{E}$ .  $\square$

**Corollary 4.21.** *Suppose CH and  $\alpha \leq \aleph_2$ . Let  $\alpha$  be a limit ordinal and let  $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{M}(\mathcal{U}_\gamma) : \gamma < \alpha, \beta \leq \alpha \rangle$  be the countable support iteration of proper iterands. Let  $\mathcal{R}_{i,\gamma}$ ,  $i = 1, \dots, k$ , be pairwise non-nearly coherent selective ultrafilters in  $\mathbf{V}_\gamma$ , increasing in  $\gamma$ . Let  $\mathcal{E}$  be a  $P$ -point in  $\mathbf{V}_{\alpha_0}$ , not nearly coherent to  $\mathcal{R}_{i,\gamma}$  in  $\mathbf{V}_\gamma$ . If each  $\Phi(\mathcal{U}_\gamma(i)) \subseteq \mathcal{R}_{i,\gamma}$  for  $\gamma < \alpha$ , then the following hold*

- (1)  $(\bar{R}, \mathcal{S}, \bar{g})$  covers at stage  $\alpha$  and  $\mathbb{P}_\alpha$  is  $(\mathcal{S}, \bar{R}, g)$ -preserving,
- (2)  $(\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma})^{+\alpha}$  is a selective coideal,
- (3) there are selective ultrafilters  $\mathcal{R}_{i,\alpha} \subseteq (\bigcup_{\gamma < \alpha} \mathcal{R}_{i,\gamma} \cup \{\mu_\gamma(i)\})^{+\alpha}$ ,  $i = 1, \dots, k$ , such that for  $i \neq j$ ,  $\mathcal{R}_{i,\alpha}$  and  $\mathcal{R}_{j,\alpha}$  are not nearly coherent and for every  $i$ ,  $\mathcal{R}_{i,\alpha}$  is not nearly coherent to  $\mathcal{E}$ .

*Proof.* (1) is proved in Lemma 4.14 and the Limit Theorem 4.18.

(2): For  $\text{cf}(\alpha) = \omega$  this is Lemma 4.19. For other limit ordinals  $\alpha \leq \omega_2$ ,  $\mathcal{R}_{i,\alpha}$  is given by (R2).

(3): This is Prop. 4.20.  $\square$

Now finally the induction is completed. As long as there are Milliken–Taylor ultrafilters  $\mathcal{U}_\gamma$  with  $\Phi(\mathcal{U}_\gamma(i)) \subseteq \mathcal{R}_{i,\gamma}$  we can carry the induction. Thus we can carry it on as long as CH holds at least. CH is used in Def. 4.11 in Theorem 2.12 and in Prop. 4.20. By name counting in the countable support iteration of proper iterands of size at most  $\aleph_1$ , we have CH in any  $\mathbf{V}_\alpha$  for  $\alpha < \omega_1$ , so we can go up to  $\mathbb{P}_{\omega_2}$ . We will see in Prop. 6.2 that we cannot go beyond  $\omega_2$ .

## 5. AN ITERATED FORCING FOR THE PROOF OF THE MAIN THEOREM

We start with a ground model  $\mathbf{V}$  that fulfils CH and  $\diamond(S_1^2)$  (and hence  $2^{\aleph_1} = \aleph_2$ ). We fix  $n \geq 2$  and let  $k = n - 1$ .

In a countable support iteration of proper forcings of iterands size  $\leq \aleph_1$  each real appears in a  $\mathbf{V}_\alpha$  for some  $\alpha$  with countable cofinality [34, Ch. III]. Recall our notation  $\mathbf{V}_\alpha = \mathbf{V}^{\mathbb{P}_\alpha}$ . A reflection property ensures that each ultrafilter  $\mathcal{W}$  in the final model has  $\omega_1$ -club many  $\alpha \in \omega_2$  such that  $\mathcal{W} \cap \mathbf{V}_\alpha$  has a  $\mathbb{P}_\alpha$ -name and is an ultrafilter in  $\mathbf{V}_\alpha$  (see [11, Item 5.6 and Lemma 5.10]). A subset of  $\omega_2$  is called  $\omega_1$ -club if it is unbounded in  $\omega_2$  and closed under suprema of strictly ascending sequences of lengths  $\omega_1$ . A subset of  $\omega_2$  is called  $\omega_1$ -stationary if it has non-empty intersection with every  $\omega_1$ -club. By well-known techniques based on coding  $\mathbb{P}_\alpha$ -names for filters as subsets of  $\omega_2$  (e.g., such a coding is carried out in [28, Claim 2.8]) and based on the maximal principle (see, e.g., [19, Theorem 8.2]) the  $\diamond(S_1^2)$ -sequence  $\langle \mathcal{S}_\alpha : \alpha \in S_1^2 \rangle$  gives  $\omega_1$ -club often a  $\mathbb{P}_\alpha$ -name  $\mathcal{S}_\alpha$  for an ultrafilter in  $\mathbf{V}_\alpha$  such

that for any ultrafilter  $\mathcal{W} \in V^{\mathbb{P}_{\omega_2}}$  there are  $\omega_1$ -stationarily many  $\alpha \in S_1^2$  with  $\mathcal{W} \cap V^{\mathbb{P}_\alpha} = \mathcal{S}_\alpha$ . For names  $\mathfrak{x}$  and objects  $x$  we use the rule  $\mathfrak{x}[G] = x$ . Often we write  $x$  for  $\mathfrak{x}$ , in particular for the generic reals  $\mu_\beta(i)$ ,  $i = 1, \dots, k$ , and the ultrafilters  $\mathcal{R}_{i,\alpha}$ ,  $\mathcal{S}_{i,\beta}$ ,  $\mathcal{U}_\beta$  ( $\beta < \omega_2$ ,  $\alpha \leq \omega_2$ ) and the relations  $R_{n,\beta,a}$  and  $R_{n,\beta}$  for  $\beta < \omega_2$  of our iterated forcing: the list of properties below is mostly about names.

We fix a diamond sequence  $\langle \mathcal{S}_\alpha : \alpha \in S_1^2 \rangle$ . We also fix a  $P$ -point  $\mathcal{E} \in \mathbf{V}$  that will be preserved throughout our iteration.

We construct by induction on  $\alpha \leq \omega_2$  a countable support iteration of proper forcings  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \omega_2, \alpha \leq \omega_2 \rangle$  and simultaneously sequences of  $\mathbb{P}_\beta$ -names  $\mathcal{S}_{j,\beta}$ ,  $i, j \in \{1, \dots, k\}$ ,  $\mathcal{U}_\beta$ ,  $\beta \leq \omega_2$  and  $\mathbb{P}_\gamma$ -names  $\mathcal{R}_{i,\gamma}$ ,  $i, j \in \{1, \dots, k\}$ ,  $\bar{\mathcal{R}}_\gamma$ ,  $\bar{g}_{\gamma,a}$  such that for any  $\alpha \leq \omega_2$ , the initial segment  $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{U}_\beta, \mathcal{R}_{i,\gamma}, \mathcal{S}_{j,\beta}, \bar{\mathcal{R}}_\gamma, \bar{g}_{\gamma,a} : \beta < \alpha, \gamma \leq \alpha, i, j \in \{1, \dots, k\} \rangle$  fulfils:

(P1) For all  $\beta < \alpha$ ,

$\Vdash_{\mathbb{P}_\beta} \text{“}\mathbb{Q}_\beta = \text{M}(\mathcal{U}_\beta) \text{ for a Milliken–Taylor ultrafilter } \mathcal{U}_\beta,$   
 adding  $(\mu_\beta(i) : i = 1, \dots, k)$

with  $(\hat{\min}_k(\mathcal{U}_\beta), \hat{\max}_k(\mathcal{U}_\beta)) = \bigotimes_{i=1}^k \mathcal{R}_{i,\beta} \otimes \bigotimes_{j=1}^k \mathcal{S}_{j,\beta}$  and

$\Phi(\mathcal{U}_\beta) \not\leq_{\text{RB}} \mathcal{E}$ ”

(P2)  $\mathbb{P}_\alpha$  is proper and

$\mathbb{P}_\alpha \Vdash \text{“}\text{fil}(\mathcal{E})^\alpha \text{ is an ultrafilter”}.$

(P3)

$\mathbb{P}_\alpha \Vdash (\forall \gamma < \alpha) (\forall i \in \{1, \dots, k\}) \left( \mathcal{R}_{i,\gamma} \subseteq \mathcal{R}_{i,\alpha} \wedge \mu_\gamma(i) \in \mathcal{R}_{i,\alpha} \text{ and } \right.$   
 $\left. \mathcal{R}_{i,\alpha} \text{ is a selective ultrafilter that is not nearly coherent to } \mathcal{E} \right)$ ”.

For  $i \neq j$ ,  $i, j = 1, \dots, k$ :

$\mathbb{P}_\alpha \Vdash \text{“} \mathcal{R}_{i,\alpha} \text{ is not nearly coherent with } \mathcal{R}_{j,\alpha} \text{”}.$

(P4) If  $\beta \in S_1^2 \cap \alpha$  and if  $\mathcal{S}_\beta$  is a  $\mathbb{P}_\alpha$ -name  $\mathcal{W}$  for an ultrafilter in  $\mathbf{V}^{\mathbb{P}_\beta}$  and in  $\mathbf{V}^{\mathbb{P}_\beta}$ ,  $\mathcal{W}$  is not nearly coherent to any of the  $\mathcal{R}_{i,\beta}$ ,  $i = 1, \dots, n-1$ , then  $\Vdash_{\mathbb{P}_{\beta+1}} \text{“}\hat{f}_\beta(\mathcal{W}) = \hat{f}_\beta(\text{cl}(\mathcal{E})) \text{”}.$  Recall,  $f_{\mu_\beta}$  is defined from  $\mu_\beta$  as above.

(P5)  $\mathbb{P}_\alpha$  is  $(\bar{\mathcal{R}}, \mathcal{S}, \bar{g})$ -preserving.

This ends the list of desired properties.

Note that in our construction  $\langle \mu_\alpha(i) : \alpha < \omega_1 \rangle$  is not a descending sequence, and it does not generate together with  $\mathcal{R}_{i,0}$  the ultrafilter  $\mathcal{R}_i$ , since in the limit steps of countable cofinality the Mathias construction from the proof of Prop. 4.20 adds  $\aleph_1$  sets. The filters  $\mathcal{S}_{j,\beta}$  are not increasing and there is no  $\mathcal{S}_{i,\omega_2}$ ,  $\mathcal{U}_{\omega_2}$ . Any  $\mathcal{S}_{i,\alpha}$  will be nearly coherent to  $\mathcal{R}_{i,\alpha}$  via  $f_{\mu_\alpha(i)}$

after stage  $\alpha + 1$  of the iteration. The  $\mathcal{S}_{i,\beta}$  are just constructed ad hoc in order to find  $\mathcal{U}_\beta$ . Also the  $\mathcal{U}_\beta$  are not increasing.

We first show that the existence of such an iteration implies our main theorem. We recall the next function:

Now we prove a lemma about small test families, extending [2, Prop 19].

**Lemma 5.1.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two ultrafilters over  $\omega$  that are both not nearly coherent to an ultrafilter  $\mathcal{E}$  over  $\omega$ . If  $\mathcal{V}$  is nearly coherent to  $\mathcal{W}$ , then there is  $E \in \mathcal{E}$  such that  $f_E(\mathcal{V}) = f_E(\mathcal{W})$ .*

*Proof.* Assume that  $h(\mathcal{V}) = h(\mathcal{W})$ . We let for  $f, g: \omega \rightarrow \omega$ ,  $f \leq_{\mathcal{W}} g$  iff  $\{n : f(n) \leq g(n)\} \in \mathcal{W}$ .  $D \subseteq {}^\omega\omega$  is called  $\leq_{\mathcal{W}}$ -dominating iff  $(\forall f \in {}^\omega\omega)(\exists g \in D)f \leq_{\mathcal{W}} g$ . Blass and Banach showed [2, Prop 19]  $\{\text{next}(\cdot, E) : E \in \mathcal{E}\}$  is  $\leq_{\mathcal{V}}$ -dominating and  $\leq_{\mathcal{W}}$ -dominating. We let  $h^{-1}[\{i\}] = [\pi_i, \pi_{i+1})$  and  $f_h(n) = \pi_{i+1}$  for  $h(n) = i$ . We take  $E \in \mathcal{E}$  such that  $f_h \leq_{\mathcal{V}, \mathcal{W}} \text{next}(\cdot, E)$ , witnessed by  $V \in \mathcal{V}$  and  $W \in \mathcal{W}$ . Claim:  $f_E(\mathcal{V}) = f_E(\mathcal{W})$ . Let  $V_1 \in \mathcal{V}$ ,  $V_1 \subseteq V$  and let  $W_1 \in \mathcal{W}$ ,  $W_1 \subseteq W$ . Take  $i = h(v) = h(w)$  for some  $v \in V_1$ ,  $w \in W_1$ . Then  $v, w \in [\pi_i, \pi_{i+1})$  and  $\pi_{i+1} = f_h(v) = f_h(w) \leq \text{next}(v, E)$  and  $\leq \text{next}(w, E)$ , since  $v \in V_1 \subseteq V$ ,  $w \in W_1 \subseteq W$ . So  $f_E(v) = f_E(w)$ .  $\square$

**Definition 5.2.**  $\mathcal{R}_i = \bigcup_{\alpha < \omega_2} \mathcal{R}_{i,\alpha}$ .

**Lemma 5.3.** *Assume that  $\mathbb{P}$  has the properties listed above. Then in  $\mathbf{V}_{\omega_2}$  there are exactly the near coherence classes of  $\mathcal{E}$  and of  $\mathcal{R}_i$ ,  $i = 0, \dots, n-1$ .*

*Proof.* By properness our iteration preserves  $\aleph_1$ . It preserves  $\aleph_2$ , because any collapse would appear at some intermediate step  $\mathbb{P}_\alpha$ , but  $\mathbb{P}_\alpha$  has size  $\aleph_1$  and the  $\aleph_2$ -c.c., see [1, Theorem 2.10, Theorem 2.12]. So  $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{V}^{\mathbb{P}}}$  and  $\aleph_2^{\mathbf{V}} = \aleph_2^{\mathbf{V}^{\mathbb{P}}}$ , and in the following we just write  $\aleph_1, \aleph_2$ .

Let  $\mathcal{W} \in \mathbf{V}_{\omega_2}$  be any ultrafilter over  $\omega$ .

There are  $\omega_1$ -club many  $\alpha \in S_1^2$  such that  $\mathcal{W} \cap \mathbf{V}_\alpha$  is an ultrafilter in  $\mathbf{V}_\alpha$ , denote this club by  $C_{\mathcal{W}}$ . Let  $S_{\mathcal{W}} \subseteq C_{\mathcal{W}}$  be a stationary subset on which  $\mathcal{W} \cap \mathbf{V}_\alpha = D_\alpha$ , the ultrafilter given by the diamond.

First case: There is an  $\alpha \in S_{\mathcal{W}}$  such that  $\mathcal{W} \cap \mathbf{V}_\alpha$  is not nearly coherent to any of the  $\mathcal{R}_{i,\alpha}$ ,  $i = 1, \dots, k$ . Then according to item (P4),  $f_{\mu_\alpha}(\mathcal{E}) = f_{\mu_\alpha}(D_\alpha) = f_{\mu_\alpha}(\mathcal{W} \cap \mathbf{V}_\alpha)$  is ultra, and hence  $f_{\mu_\alpha}(\mathcal{W} \cap \mathbf{V}_\alpha) = f_{\mu_\alpha}(\mathcal{W}) = f_{\mu_\alpha}(\mathcal{E})$ .

Second case: For any  $\alpha \in S_{\mathcal{W}}$  there is an  $i_\alpha \in \{1, \dots, k\}$  and a finite-to-one function  $h_\alpha \in \mathbf{V}_\alpha$  such that  $h_\alpha(\mathcal{W} \cap \mathbf{V}_\alpha) = h_\alpha(\mathcal{R}_{i_\alpha, \alpha})$ . Then  $\mathcal{W} \cap \mathbf{V}_\alpha$  is not nearly coherent to  $\mathcal{E}$  since otherwise  $\mathcal{R}_{i_\alpha, \alpha}$  would be nearly coherent to  $\mathcal{E}$ . Now by Lemma 5.1 on small test families there is a cofinal set  $C \subseteq S_{\mathcal{W}}$  and there is an  $i \in \{1, \dots, k\}$  and there is an  $E \in \mathcal{E} \cap \mathbf{V}_0$  such that for any  $\alpha \in C$   $f_E(\mathcal{R}_{i,\alpha}) = f_E(\mathcal{W} \cap \mathbf{V}_\alpha)$ . But then  $f_E(\mathcal{R}_i) = f_E(\mathcal{W})$ .  $\square$

Now we define by induction on  $\alpha \leq \omega_2$  an iteration with the desired properties (P1) to (P5). First we consider the successor of an ordinal.

**Lemma 5.4.** *Assume that  $\alpha \in \omega_2$  and that  $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{U}_\beta, \mathcal{R}_{i,\gamma}, \mathcal{S}_{i,\beta}, \bar{R}_\gamma, \bar{g}_{\gamma,a} : \gamma \leq \alpha, \beta < \alpha, i = 1, \dots, k \rangle$  is defined with the properties (P1) to (P5). Then there is a  $\sigma$ -centred  $\mathbb{Q}_\alpha$  such that  $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{U}_\beta, \mathcal{R}_{i,\gamma}, \mathcal{S}_{i,\beta}, \bar{R}_\gamma, \bar{g}_{\gamma,a} : \gamma \leq \alpha + 1, \beta < \alpha + 1, i = 1, \dots, k \rangle$  has properties (P1) to (P5).*

*Proof.* Let  $G_\alpha \subseteq \mathbb{P}_\alpha$  be generic over  $\mathbf{V}$  and let  $G_\beta = \mathbb{P}_\beta \cap G_\alpha$  for  $\beta < \alpha$ .

By induction hypothesis we have in  $\mathbf{V}_\alpha$   $\mathcal{R}_{i,\alpha}$  with property (P3). Given  $\mathcal{E}$  and (handed down by the diamond in case  $\text{cf}(\alpha) = \omega_1$ )  $\mathcal{V}$  (that is handed down by the diamond in case  $\text{cf}(\alpha) = \omega_1$ , otherwise let  $\mathcal{V} = \mathcal{E}$ ) we choose selective  $\mathcal{S}_j$  as in Lemma 2.30 that are not nearly coherent to  $\mathcal{E}, \mathcal{V}$ . Then we use Theorem 2.12 to get  $\mathcal{U}_\alpha$ . Then we take  $\mathbb{Q}_\alpha = \mathbb{M}(\mathcal{U}_\alpha)$ . Then we have (P1) for  $\alpha + 1$ . Now  $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$  has by the preservation of properness in successor steps and by Theorem 3.5 property (P2). Now by Theorem 4.5 we have  $\text{fil}(\mathcal{R}_{i,\alpha} \cup \{\mu_\alpha(i)\})^{\mathbb{M}(\mathcal{U}_\alpha)}$  is a selective ultrafilter that is not  $\leq_{\text{RB}} \mathcal{E}$  and for  $j \neq i$ ,  $\text{fil}(\mathcal{R}_{i,\alpha} \cup \{\mu_\alpha(i)\})^{\mathbb{M}(\mathcal{U}_\alpha)}$  and  $\text{fil}(\mathcal{R}_{j,\alpha} \cup \{\mu_\alpha(j)\})^{\mathbb{M}(\mathcal{U}_\alpha)}$  are not nearly coherent. So we have (P3) for  $\alpha + 1$ . In case of  $\text{cf}(\alpha) = \omega_1$  and that  $\mathcal{W}$  is handed down by the diamond, we use [23, Theorem 4.5] that  $\mathbb{M}(\mathcal{U}_\alpha)$  forces that  $f_{\mu_\alpha}(\mathcal{E}) = f_{\mu_\alpha}(\mathcal{W})$ . So we have (P4) for  $\alpha + 1$ . (P5) follows from Theorem 4.15.  $\square$

**Lemma 5.5.** *Let  $\alpha \in \omega_2$  be the limit ordinal of countable cofinality. If for each  $n$ ,  $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{U}_\beta, \mathcal{R}_{i,\gamma}, \mathcal{S}_{i,\beta}, \bar{R}_\gamma, \bar{g}_{\gamma,a} : \beta < \alpha_n, \gamma \leq \alpha_n, i = 1, \dots, k \rangle$  fulfils (P1) to (P5), then also the countable support limit  $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{U}_\beta, \mathcal{R}_{i,\gamma}, \mathcal{S}_{i,\beta}, \bar{R}_\gamma, \bar{g}_{\gamma,a} : \beta < \alpha, \gamma \leq \alpha, i = 1, \dots, k \rangle$  fulfils (P1) to (P5).*

*Proof.* (P1) is inherited from the induction hypothesis. For (P2) we use a well-known preservation theorem: The countable support limit of forcings preserves each  $P$ -point that is preserved by all approximations [11, Theorem 4.1]. For the first half of (P2) we use that the countable support limit of proper forcings is proper [35, III, 3.2]. There are no new instances of (P4) in steps of countable cofinality. The properties (P3), (P5) follow from Corollary 4.21 and so we have the conclusion of Lemma 4.19. Now under this conclusion we use Prop. 4.20 to construct  $\mathcal{R}_{i,\alpha}$ .  $\square$

**Lemma 5.6.** *Let  $\alpha = \lim_{\varepsilon < \mu} \alpha_\varepsilon$  be the limit of a strictly increasing sequence of ordinals in  $\omega_2$  and let  $\mu$  be  $\omega_1$  or  $\omega_2$ . If for all  $\varepsilon$ ,  $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{U}_\beta, \mathcal{R}_{i,\gamma}, \mathcal{S}_{i,\beta}, \bar{R}_\gamma, \bar{g}_{\gamma,a} : \beta < \alpha_\varepsilon, \gamma \leq \alpha_\varepsilon, i = 1, \dots, k \rangle$  fulfils (P1) to (P5), then also  $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{U}_\beta, \mathcal{R}_{i,\gamma}, \mathcal{S}_{i,\beta}, \bar{R}_\gamma, \bar{g}_{\gamma,a} : \beta < \alpha, \gamma \leq \alpha, i = 1, \dots, k \rangle$  fulfils (P1) to (P5).*

*Proof.* (P1) and (P4) are inherited from the induction hypothesis. For (P2) we invoke the same citations as in the previous lemma (or we use directly that no real enters at a step of cofinality  $\omega_1$ ). We let  $\mathcal{R}_{i,\alpha} = \bigcup_{\beta < \alpha} \mathcal{R}_{i,\beta}$ . (P3) is true, since every finite-to-one  $f$  with  $f(\mathcal{R}_{i,\alpha}) = f(\mathcal{E})$  or with  $f(\mathcal{R}_{i,\alpha}) = (\mathcal{R}_{j,\alpha})$  would be in a  $\mathbf{V}_\beta$  for  $\beta < \alpha$ . However  $\mathcal{R}_{i,\beta} = \mathcal{R}_{i,\alpha} \cap \mathbf{V}_\beta$  is in  $\mathbf{V}_\beta$  not nearly coherent to  $\mathcal{E}$  and not nearly coherent to  $\mathcal{R}_{j,\beta}$  for  $i \neq j$  for cofinally

many  $\beta < \alpha$ . (P5) holds for  $\alpha$  of uncountable cofinality since every real has a  $\mathbb{P}_\beta$ -name for a successor ordinal  $\beta$  or a limit  $\beta$  of countable cofinality.  $\square$

So we have proved the main theorem.

## 6. AN ANSWER TO A TECHNICAL QUESTION

We say “ $s$  diagonalises  $\mathcal{W}$ ” if for any  $W \in \mathcal{W}$ ,  $s \subseteq^* W$ . We say “we diagonalise  $\mathcal{W}$ ” if we add in a forcing extension a set  $s$  that diagonalises  $\mathcal{W}$ . The following follows from Fract 1.2: If a forcing diagonalises two non-nearly coherent ultrafilters, then it adds a dominating real and hence destroys  $\mathcal{E}$ . In our construction it is very important that we destroy the ultrafilters  $\mathcal{R}_{i,\alpha}$  but do not diagonalise them.

*Remark 6.1.* Let  $\mathbb{P}_\alpha$  be a forcing with properties (P1) to (P5). At any  $\alpha < \omega_2$ , in  $\mathbf{V}_\alpha$  there are infinitely (and hence by Banach and Blass  $2^{\omega_1}$ ) many near coherence classes of ultrafilters in  $\{\mathcal{W} : \mathcal{W} \supseteq \Phi(\mathcal{U}_\alpha(i))\}$  for any  $i = \{1, \dots, k\}$ .

Proof: If there were for some  $i = 1, \dots, k$  only finitely many, say  $\mathcal{W}_0, \dots, \mathcal{W}_r$ ,  $r \geq 1$ , representing the near coherence classes in  $\Phi(\mathcal{U}(i))$ , then there are “separators”  $W_j \in \mathcal{W}_j \setminus \bigcup_{j' \in r \setminus \{j\}} \mathcal{W}_{j'}$  and hence for one suitable  $j_0, j_1 \in \{0, \dots, r\}$   $\mu_\alpha(i) \cap W_{j_0}$  would diagonalise  $\mathcal{R}_{i,\alpha}$  and  $\mu_\alpha(i) \cap W_{j_1}$  would diagonalise  $\mathcal{S}_{i,\alpha}$ . Since  $\mathcal{R}_{i,\alpha}$  is not nearly coherent to  $\mathcal{S}_{i,\alpha}$ , we would add a dominating function.  $\square$

Finally we answer the last question in Blass’ list of open questions in [6]:

**Proposition 6.2.** *In  $\mathbf{V}^{\mathbb{P}_{\omega_2}}$  there is no Milliken–Taylor ultrafilter, even not for  $k \geq 2$ .*

*Proof.* For  $k = 1$ : The minimum and the maximum projection need to be two non nearly coherent selective ultrafilters. For  $k = 1$  we have only one. For arbitrary  $k$ . Suppose  $\mathcal{U}$  is an Milliken–Taylor ultrafilter. Then there are  $i \neq j$  such that the  $\hat{\min}(\mathcal{U})$  is nearly coherent to  $\mathcal{R}_i$  and  $\hat{\max}(\mathcal{U})$  is nearly coherent to  $\mathcal{R}_j$ .  $\Phi(\mathcal{U}) \not\leq_{\text{RB}} \mathcal{E}$  since  $\Phi(\mathcal{U})$  has character  $\aleph_2$ . Now the forcing  $\mathbb{M}(\mathcal{U})$  in  $\mathbf{V}_{\omega_2}$  is an impossible object: If we force with it we preserve  $\mathcal{E}$  according to Eisworth’ Theorem 3.5. However,  $\mathbb{M}(\mathcal{U})$  diagonalises  $\mathcal{R}_i$  and  $\mathcal{R}_j$  according to Remark 6.1, so adds a dominating real and hence destroys every ultrafilter.  $\square$

**Corollary 6.3.** *The existence of two non-isomorphic selective ultrafilters does not imply the existence of a Milliken–Taylor ultrafilter.*

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